Univariate Self-Starting Shiryaev (U3S): A Bayesian Online Change Point Model for Short Runs

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- Statistical Process Control/Monitoring (SPC/M) is an effective area of Statistics that includes all methods that deal with the **quick and valid detection of any disorder** in an ongoing process. Its main aim is to detect when a process deteriorates from its **In Control (IC)** state to the **Out Of Control (OOC)** state

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- SPC/M is applied in a **plethora of disciplines**, like:.

  industrial processes,

  medical laboratories,

  economics,

  geophysics etc.

- The **type of shifts**, i.e. the OOC states, that are most often considered in practice, are:

    Transient shifts: an isolated unusual value, i.e. an outlier. It is typically of large size.

    Persistent shifts: systematic changes to at least one parameter of a procedure , e.g. step changes, scale shifts, linear trends, rotations etc.. It is typically of small/medium size.

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- The majority of the proposed SPC/M methods, which are designed to efficiently detect them, typically requires two phases (I/II).

  Phase I is the training and typically offline phase, where independent IC data are gathered and the goal is to perform calibration of the monitoring scheme.

  Phase II follows and it is the testing and typically online phase, where new observations are collected and compared against the IC standards that established in phase I.

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- Furthermore, the phase I/II separation has certain **restrictions**.

  In phase I analysis a **large amount of independent IC samples** is needed to provide (offline) reliable estimates of the unknown parameter(s).

  The **estimation error** for the parameter(s) of interest is typically not taken into account.

  The **IC information**, which is available from phase II data, **is wasted** using one-off plugged in phase I estimates.

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Definition

A control chart will be called as self-starting if:

- it can provide testing, without the need of a preliminary training phase,
- it allows monitoring and inference after each incoming data point becomes available (online) and not retrospectively (offline),
- the IC and the OOC states contain at least one unknown parameter.

From now on, we will characterize a method as self-starting based on the above definition.

Research goal

In this work the focus is placed on:

- **individual univariate** short horizon data,
- the online detection of **persistent disorders** and the reliable **inference** for the unknown process parameter(s),
- adopting the **Bayesian perspective**, without the requirement of any calibration phase (self-starting).

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Our proposal:

- relaxes the strict assumption of known parameters,
- utilizes prior information (if available),
- focuses on detecting change points,
- provides posterior inference for all parameters of interest.

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- 3S is a generalization of the Shiryaev's process (Shiryaev, 1963) and it is based on the posterior marginal probability of a change point occurrence.
- We will provide all the assumptions and the methodological framework to handle univariate (U3S) data with changes in the mean or the variance.

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The likelihood will be:

$$f\left(\boldsymbol{x_n}|\boldsymbol{\theta},\boldsymbol{\phi},\tau\right) = \begin{cases} f\left(\boldsymbol{x_n}|\boldsymbol{\theta},\boldsymbol{\phi},\tau \leq n\right) = \prod_{i=1}^{\tau-1} f\left(x_i|\boldsymbol{\theta}\right) \prod_{i=\tau}^{n} f\left(x_i|g(\boldsymbol{\theta},\boldsymbol{\phi})\right) & \text{if } \tau \leq n \\ f\left(\boldsymbol{x_n}|\boldsymbol{\theta},\tau > n\right) = \prod_{i=1}^{n} f\left(x_i|\boldsymbol{\theta}\right) & \text{if } \tau > n \end{cases}$$

Self-starting Shiryaev (3S)

The stopping time is based on the posterior marginal probability of a change point occurrence, which is:

$$
\begin{aligned}
p(\tau \leq n | \boldsymbol{x_n}) &= \frac{f(\boldsymbol{x_n}|\tau \leq n)\,\pi(\tau \leq n)}{f(\boldsymbol{x_n}|\tau \leq n)\,\pi(\tau \leq n) + f(\boldsymbol{x_n}|\tau > n)\,\pi(\tau > n)} \\
&= \frac{\displaystyle\sum_{k=1}^{n} \frac{\pi(\tau = k)}{\pi(\tau > n)} \cdot BF_{k,n+}}{\displaystyle\sum_{k=1}^{n} \frac{\pi(\tau = k)}{\pi(\tau > n)} \cdot BF_{k,n+} + 1}
\end{aligned}
$$

where $BF_{k,n+} = \dfrac{f(\boldsymbol{x_n}|\tau = k)}{f(\boldsymbol{x_n}|\tau > n)}$ (Bayes Factor), compares the evidence the $k^{th} \leq n$ observation to be the change point against the evidence all available n observations to be IC.

Self-starting Shiryaev (3S)

- The marginal distributions involved in the computation are:

$$f(\boldsymbol{x_n}|\tau > n) = \int_{\Theta} f(\boldsymbol{x_n}|\boldsymbol{\theta}, \tau > n)\pi(\boldsymbol{\theta})d\boldsymbol{\theta}$$

$$f(\boldsymbol{x_n}|\tau \leq n) = \int_{\Phi}\int_{\Theta} f(\boldsymbol{x_n}|\boldsymbol{\theta}, \boldsymbol{\phi}, \tau \leq n)\pi(\boldsymbol{\theta})\pi(\boldsymbol{\phi})d\boldsymbol{\theta}d\boldsymbol{\phi}$$

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- If the prior $\pi(\boldsymbol{\theta})$ is improper, we sacrifice the s first observations $\boldsymbol{x_{1:s}}$ necessary to make the posterior $p(\boldsymbol{\theta}|\boldsymbol{x_{1:s}})$ proper and use it instead of $\pi(\boldsymbol{\theta})$.

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Constant decision limit p^*

$T(p^*) = inf\left\{n \geq 1 : p\left(\tau \leq n|\boldsymbol{x_n}\right) \geq p^*\right\}$

Adapted decision limit p_n^*

$$T(p_n^*) = inf\left\{ n \geq 1 : p\left(\tau \leq n|\boldsymbol{x_n}\right) \geq p_n^* = \frac{K \cdot \sum_{k=1}^{n} \frac{\pi(\tau = k)}{\pi(\tau > n)}}{K \cdot \sum_{k=1}^{n} \frac{\pi(\tau = k)}{\pi(\tau > n)} + 1} \right\}$$

where p^* and K are chosen with respect to the false alarm tolerance.

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- Apart from change point detection, we can also provide inference for the unknown parameters:

  - $\begin{cases} p\left(\boldsymbol{\theta}|\boldsymbol{x_n}\right) & \text{if a change point did not occur} \\ p\left(\boldsymbol{\theta}, \boldsymbol{\phi}, \tau|\boldsymbol{x_n}\right) & \text{an alarm is raised} \end{cases}$

Self-starting Shiryaev (3S)

IC scenario ($\tau > n$)

OOC scenario ($\tau \leq n$)

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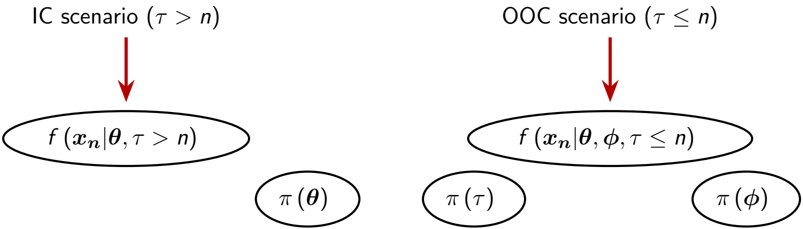

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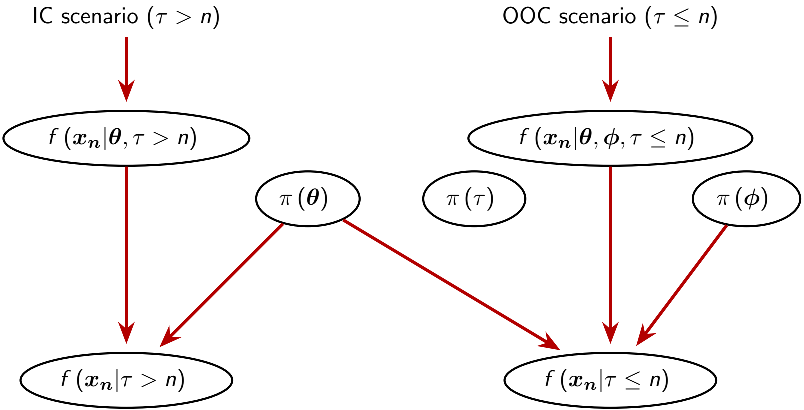

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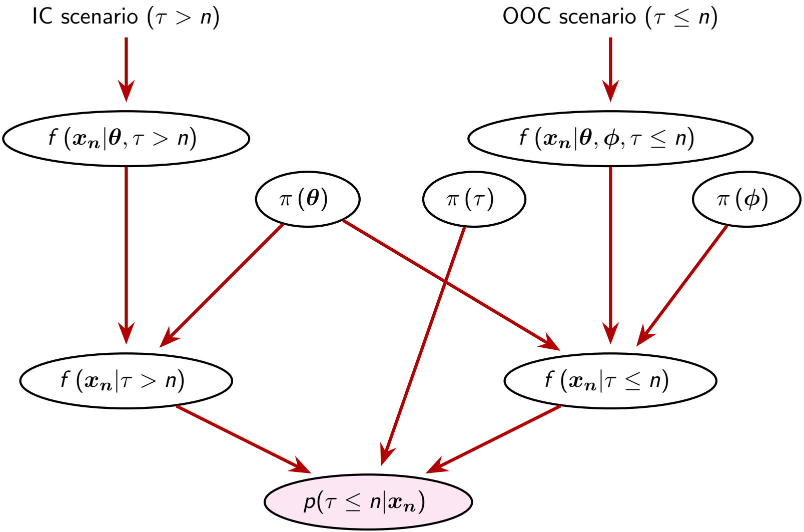

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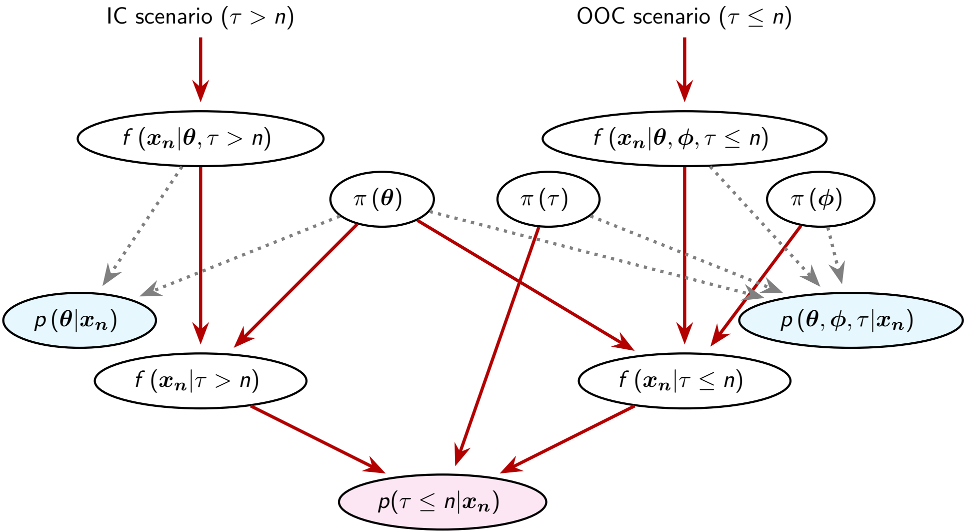

U3S for the Normal mean

- **Model parameters**

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  $\boldsymbol{\theta} = \left(\theta_1, \theta_2^2\right)$: the mean and the variance of the data

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Prior setting

- $\pi(\boldsymbol{\theta}) \propto L(\boldsymbol{\theta}|\boldsymbol{Y})^{\alpha_0}\,\pi_0(\boldsymbol{\theta})$ (power prior, Ibrahim 2000), where:
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  τ is the location of a potential change point,
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Application to real data

Data description:

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- The observations arrive sequentially, assuming:

$$\boldsymbol{X}_i|\boldsymbol{\theta} \overset{iid}{\sim} N\left(\theta_1, \theta_2^2\right)$$

Weakly informative prior setting:

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- $\pi_0\left(\theta_1, {\theta_2}^2 | \boldsymbol{\tau}\right) \sim NIG\left(31.8, 1/2, 2, 4.41\right)$

  Also, we have $n_0 = 37$ IC historical data with $\bar{\mathbf{y}} = 31.73$ and $var(\mathbf{y}) = 3.31$ $(\alpha_0 = 1/n_0)$. Combining these two sources of information, we obtain:

  $\pi\left(\theta_1, {\theta_2}^2 | \boldsymbol{Y}, \alpha_0, \boldsymbol{\tau}\right) \sim NIG\left(31.75, 3/2, 5/2, 6.02\right)$

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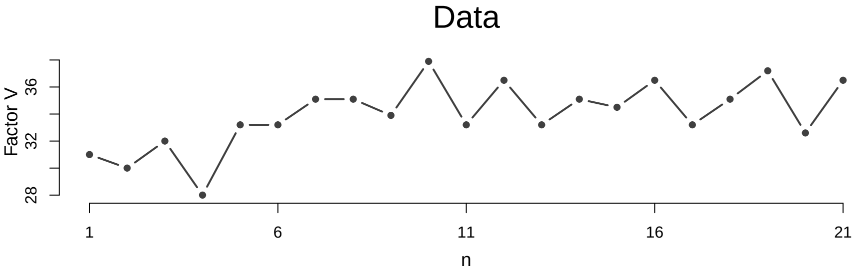
Decision limit elicitation:

- We set ${p_n}^*$ to control $PFA = 5\%$ for $n = 21$ data points.

Model illustration



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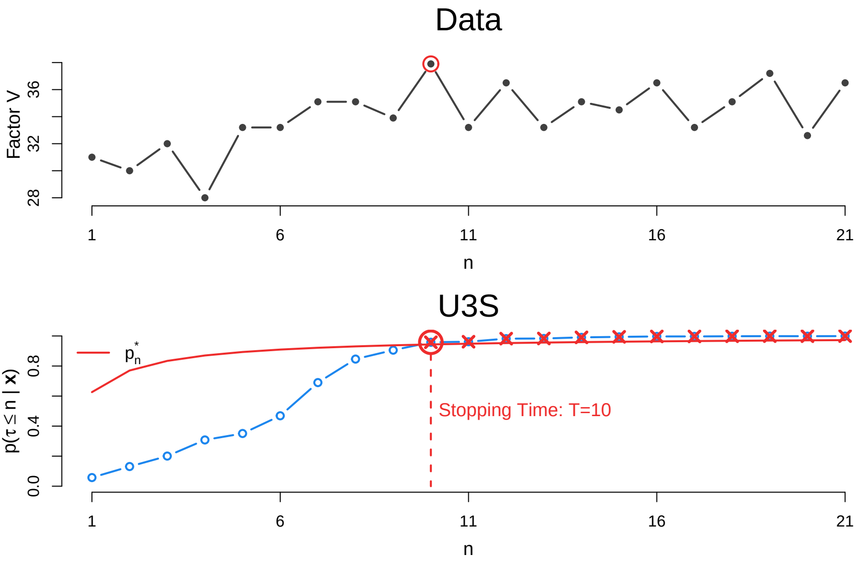

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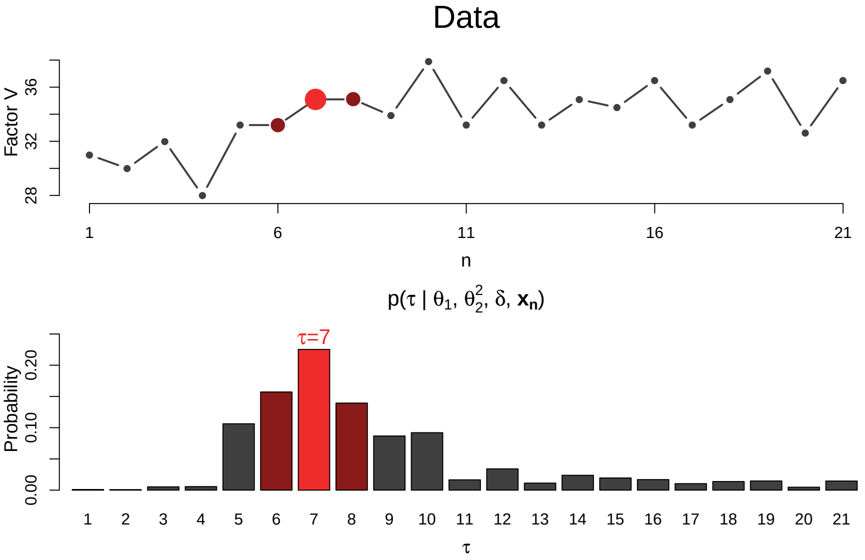

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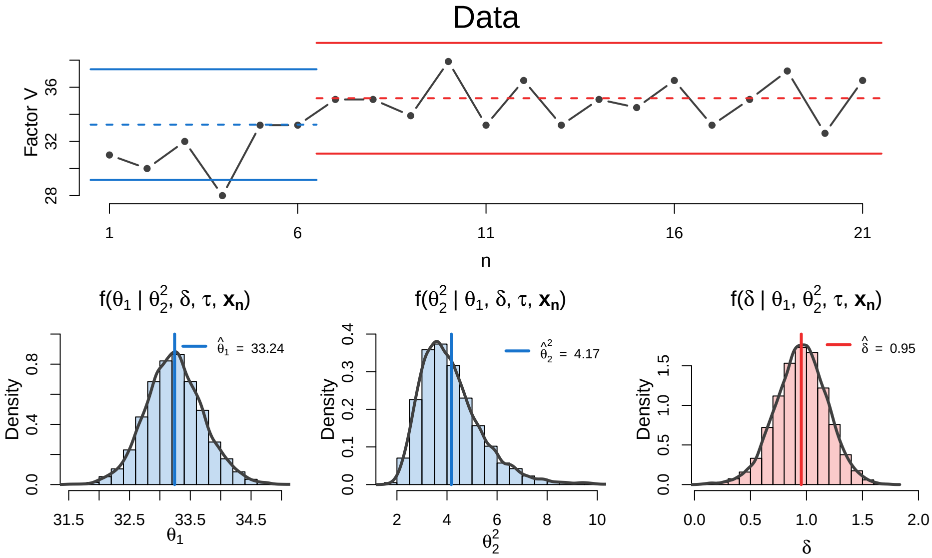

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- We are interested in a two-sided U3S, either for an inflation or a shrinkage of the variance.

U3S application to real data

Data description:

- The dataset comes from Villanueva-Guerra et al. (2017) and it refers to $n = 60$ monthly increments in the S&P 500, which is an American stock market index.
- We are interested in a two-sided U3S, either for an inflation or a shrinkage of the variance.
- The observations arrive sequentially, assuming:

$$\boldsymbol{X}_i|\boldsymbol{\theta} \overset{iid}{\sim} N\left(\theta_1, \theta_2^2\right)$$

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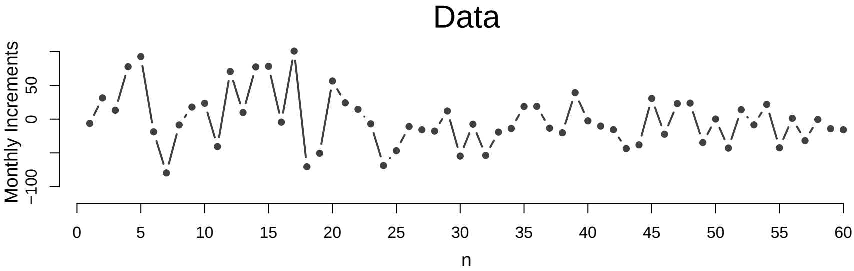
Decision limit elicitation:

- We set ${p_n}^*$ to control $PFA = 10\%$ for $n = 60$ data points.

Model illustration



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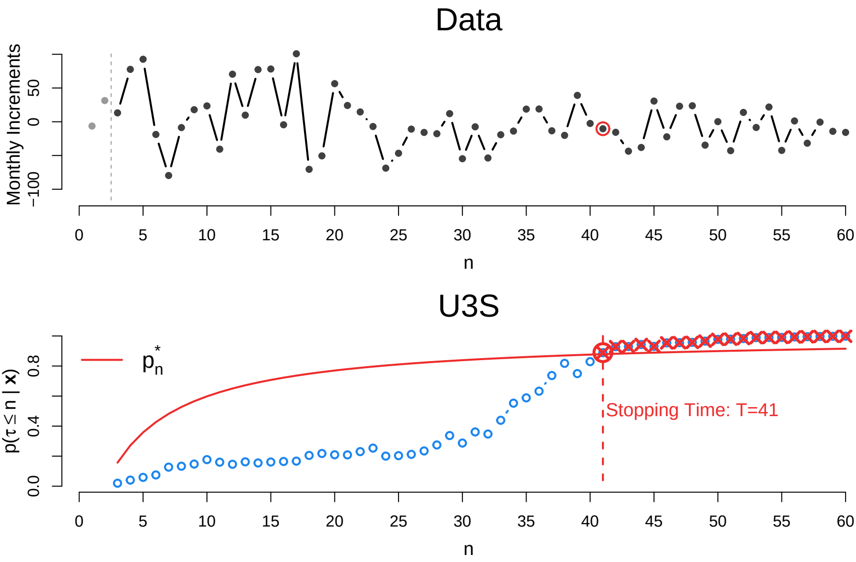

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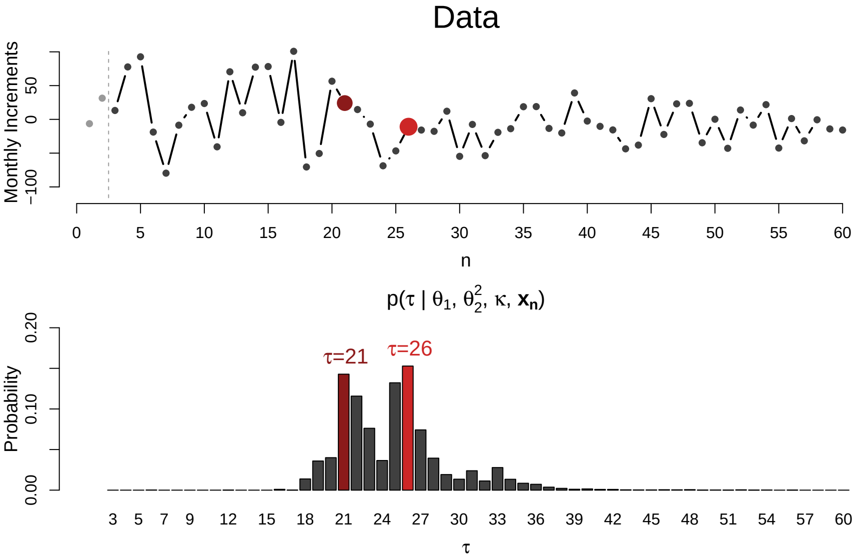

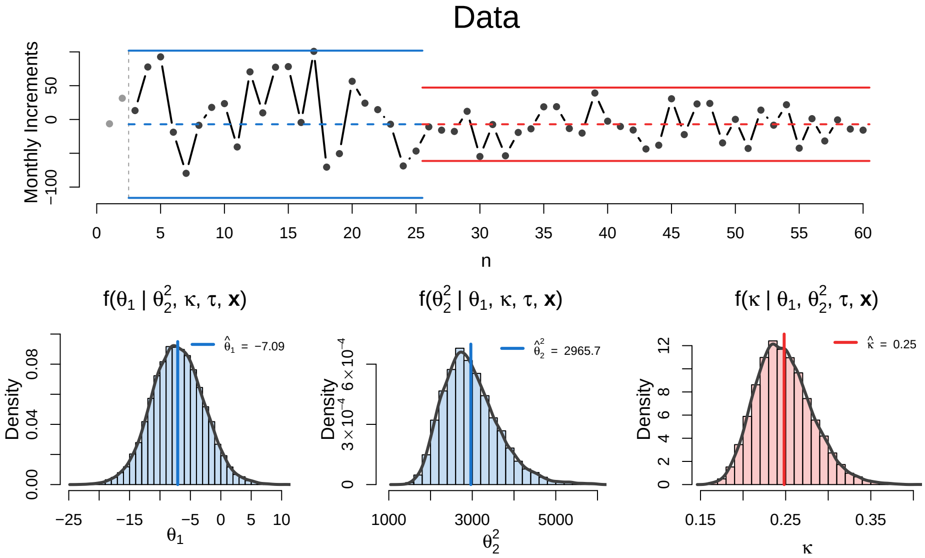
Data
Monthly Increments
−100
0
50
0
5
10
15
20
25
30
35
40
45
50
55
60
n
f(θ1 | θ2², κ, τ, x)
Density
0.00
0.04
0.08
θ̂1 = −7.09
−25
−15
−5
0
5
10
θ1
f(θ2² | θ1, κ, τ, x)
Density
0
3×10⁻⁴
6×10⁻⁴
θ̂2² = 2965.7
1000
3000
5000
θ2²
f(κ | θ1, θ2², τ, x)
Density
0
2
4
6
8
12
κ̂ = 0.25
0.15
0.25
0.35
κ

Competing methods and sensitivity analysis

Competing methods:

- Self-Starting CUSUM (SSC, Hawkins and Olwell, 1998),
- Recursive Segmentation and Permutation (RS/P, Capizzi and Masarotto, 2013),
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IC data:

- **Mean**: For $N = 50$, we assume $X_i | \left(\theta_1, \theta_2^2\right) \overset{i.i.d.}{\sim} N\left(\theta_1, \theta_2^2\right)$, where $\theta_1 = 0$ and $\theta_2^2 = 1$. We simulate $10,000$ iterations of each random sample.
- **Variance**: For $N = 50$, we assume $X_i | \left(\theta_1, \theta_2^2\right) \overset{i.i.d.}{\sim} N\left(\theta_1, \theta_2^2\right)$, where $\theta_1 = 0$ and $\theta_2^2 = 1$. We simulate $10,000$ iterations of each random sample.

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OOC scenarios:

- **Mean**: Step changes for the mean from a $N(1, 1)$ and initiating at location 11, or 26, or 41.
- **Variance**: 50% sd inflation shift, i.e. the OOC is $N(0, 1.5)$, initiating at location 11, or 26, or 41.

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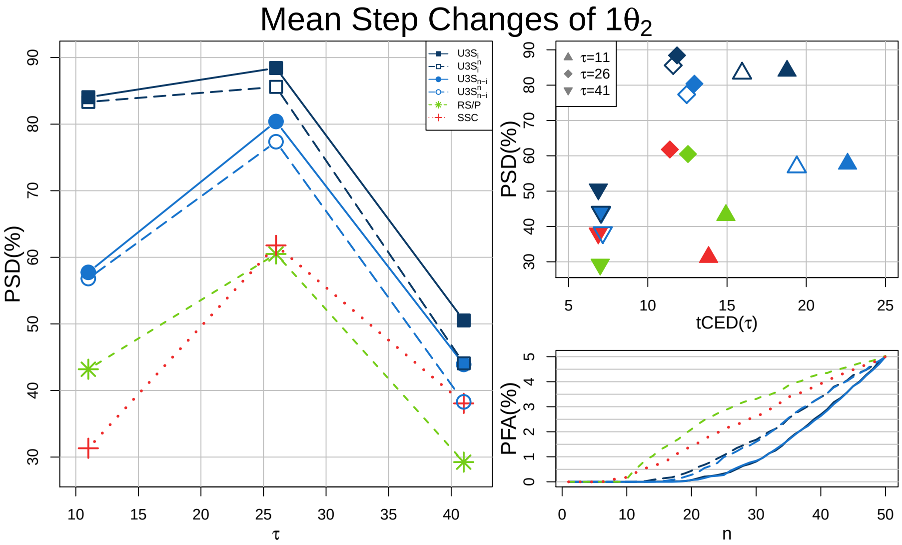
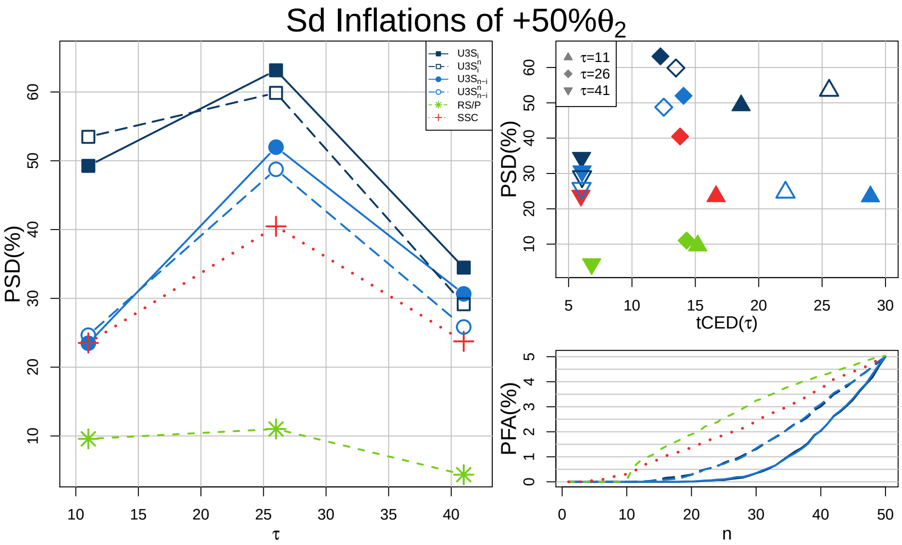
- We estimate the truncated Conditional Expected Delay (tCED)

$$tCED(\omega) = E_{\omega}(T - \omega + 1 | \omega \leq T \leq N)$$


Mean Step Changes of 1θ2
U3S_i
U3S_i^n
U3S_{n−i}
U3S_{n−i}^n
RS/P
SSC
PSD(%)
τ
τ=11
τ=26
τ=41
tCED(τ)
PFA(%)
n


Sd Inflations of +50%θ2
U3S_i
U3S_i^n
U3S_n−i
U3S_n−i^n
RS/P
SSC
PSD(%)
τ
τ=11
τ=26
τ=41
tCED(τ)
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n

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- is more resistant in absorbing an OOC scenario.

References

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Thank you!

Questions?