The difference between the weak and the strong core from the design point of view

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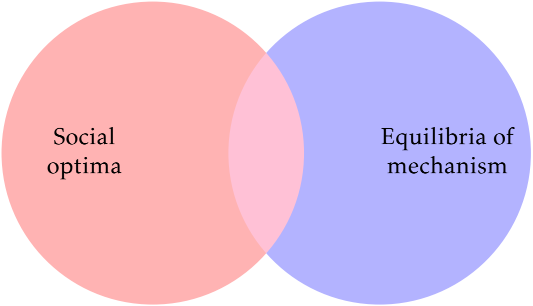
- (Full) implementation: All equilibria are socially optimal **and** each social optimum can be realized through some equilibrium.



Implementation via mechanisms vs implementation via rights structures

Mechanisms	**Rights structures**
Strategies	Rights
Noncooperative equilibrium notions	Cooperative equilibrium notions

Mechanisms and rights structures, example

Primitives

$N = \{1,2\}$, $\mathcal{R} = \{R, R'\}$

$\phi(R) = \{x, w\}$, $\phi(R') = \{x\}$

R_1	R_2	R'_1	R'_2
x	w	x	w
z	z	z	x
w	x	y	z
y	y	w	y

Table: Preferences

1/2	l	r
U	x	y
D	z	w

Table: Mechanism/game form

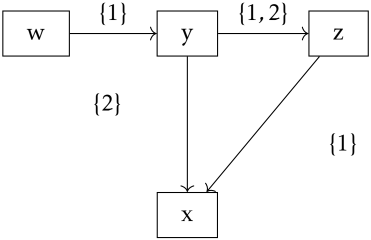


Figure: Rights structure

To cooperate or not?

R_1	R_2
xy	x
z	y
	z

Figure: Preferences

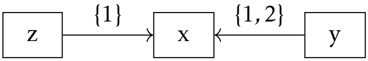


Figure: Rights structure

- Weak core: $C(\Gamma, R) = \{x, y\}$.
- Strong core: $SC(\Gamma, R) = \{x\}$.

This paper

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- What SCRs can we implement is strong core?

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- What SCRs can we implement in weak core?
- What SCRs can we implement is strong core?
- Remarks and applications.

Related literature

Noncooperative mechanisms
Maskin (1999) RES
Moore and Repullo (1990) JET
Jackson (1991) ECMA
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Rights structures
Koray and Yildiz (2018) JET
Korpela et al (2020) JET
Korpela et al (2021) GEB
Savva (2021) EL
Lombardi et al (2021)

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Interpretation: social planner wants to implement ϕ, but knows only $\mathcal{R}$.

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Strong core: $s \in SC(\Gamma, R)$ if for all $t \in S$, there is no $K \in \gamma(s,t)$, such that for all $i \in K$ we have $h(t) R_i h(s)$ with strict preference for at least one $j \in K$.

Implementation:

$$\text{for any } R \in \mathcal{R},\ \phi(R) = h(C(\Gamma, R)) \text{ (or } = h(SC(\Gamma, R))).$$

Example

Implementation in weak core

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Maskin-monotonicity

Implementation in weak core

Maskin-monotonicity

There exists $Y \subseteq X$ with $\phi(\mathcal{R}) \subseteq Y$, such that for all R, R' and $x \in \phi(R)$:

$$\text{if for all } i, [L_i(x,R) \subseteq L_i(x,R')] \cap Y, \text{ then } x \in \phi(R').$$

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$$\text{if for all } i,\ Y \subseteq L_i(x,R), \text{ then } x \in \phi(R).$$

Theorem

The following are equivalent:

(i) *ϕ is implementable in weak core.*

(ii) *ϕ satisfies Maskin-monotonicity and unanimity.*

(iii) *ϕ is implementable in weak core by an individual-based rights structure.*

Example

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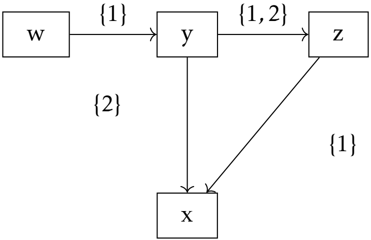

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- There is no value in *coalitional rights* from the economic design point of view.

Implementation in strong core

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A few further definitions:

$$I_K(x,R) = \cap_{i\in K}\{y \in X | xI_iy \text{ for all } i \in K\}, \text{ and}$$
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Weak SC-monotonicity

There exists $Y \supseteq \phi(\mathcal{R})$ such that for all R, R' and $x \in \phi(R)$:

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Theorem

The following are equivalent:

(i) *An SCR ϕ is implementable in strong core.*

(ii) *An SCR ϕ satisfies weak SC-monotonicity and unanimity with respect to Y.*

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  - Restoring value of coalitional rights in mechanism design.

The rationing problem

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- Set of allocations $Z \equiv \{x \in \mathbb{R}^n_+ | \sum_{i \in N} x_i = M\}$.
- For all $i \in N$, $\succsim_i \in \mathcal{L}_i$ and $x, y \in Z$:

$$xR_iy \text{ if and only if } x_i \succsim_i y_i.$$

The no-envy correspondence

No-envy correspondence

$NE : \mathcal{R} \to 2^Z$, such that, for all $x \in Z$, we have $x \in NE(R)$ if and only if for all $\{i, j\} \subseteq N, x_i \succsim_i x_j$.

We are in general interested in the intersection of no-envy with strong (or weak) Pareto.

Proposition

Suppose that $NE \cap SPO$ is non-empty. Then, it is implementable in strong core.

Single-plateaued preferences

$N \equiv \{1,2\}$, $\mathcal{R} = \{R, R'\}$, $M = 20$.

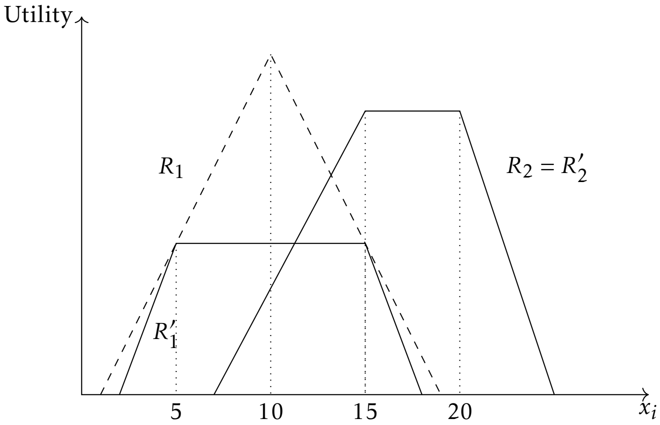

Figure: Single-plateaued preferences

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Double implementation: for all $R \in \mathcal{R}$, $\phi(R) = h(C(\Gamma, R)) = h(SC(\Gamma, R))$.

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- Importance of strong core as an equilibrium concept.

The end

Thanks! :-)

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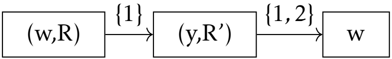

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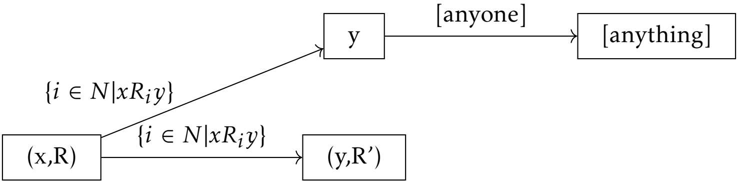

Figure: Canonical rights structure

Example