# Measuring Technical Efficiency Under Factor Nonsubstitution: A Stochastic von Liebig Crop Response Model

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The authors wish to thank Quirino Paris for kindly providing the data used in the present paper.

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**Abstract**

*The present paper develops an econometric model for measuring input-oriented technical efficiency when the underlying technology is characterized by the lack of substitution between inputs. In this instances, Farrell's radial measure of technical inefficiency is inappropriate as it may be possible to identify a technical inefficient bundle as technical efficient. Instead Russell's non-radial indices can adequately measure technical inefficiency in factor limitation models. To this end, a disequilibrium model augmented with a regime specific technical inefficiency term is proposed and its likelihood function derived together with the computation of technical efficiency under specific distributional assumptions. The framework under which the model is proposed is the well known von Liebig hypothesis that analyses crop response to different levels of fertilizer nutrients. Application of the proposed stochastic von Liebig crop response model to the experimental data of Heady and Pesek (1954) points out to the fact that technical inefficiency can arise for a subset of the nutrients considered.*

## I. Introduction

The abundant economic literature on the estimation of stochastic production frontier functions and the subsequent measurement of technical inefficiency has in general assumed that the underlying technology displays some degree of substitutability between factors of production. In effect, a production technology with zero input elasticity of substitution would imply that the cost-minimizing inputs are independent of their prices, which is an untenable assumption in many real world applications. And yet some specific types of production activities exhibit a zero elasticity of substitution. Some examples are given by Komiya (1962) who investigated the technological progress in the US steam power industry, Lau and Tamura (1972) who propose the use of a non-homothetic Leontief production function to analyze the Japanese petrochemical industry, Nakamura (1990) who utilized a nonhomothetic generalized Leontief technological structure for empirically analyzing Japanese iron and steel industry, Holvad et al. (2004) who maintain that the transport industry might be characterized by Leontief-type technologies when analyzing cost efficiency in the Norwegian bus industry. Yet in agricultural economics literature, the contributions of Paris and co-authors when modeling crop response to different fertilizer's nutrients levels they maintained zero substitution among crop nutrients..

The issue of measuring technical inefficiency in the case of the above Leontief type technologies is of interest in itself since Farrell's radial measures, which are the basis for most applied work on the measurement of efficiency, might not be adequate. Indeed, it might sometimes classify inefficient input combinations as being efficient and *vice-versa*, moreover, input- and output-oriented measures might not coincide. Once technology is governed by a Leontief type structure it is very plausible to have inefficiency being displayed by none, all or a subset of the inputs, rendering radial measures unsatisfactory. In addition, output-oriented measures might fail to recognize inefficiencies when they affect a subset of the inputs only.

In the light of the above discussion, the purpose of this paper is to propose a stochastic frontier model, where input-oriented technical efficiency can be defined and measured. Our measure of efficiency is based on Russell's measure which is a non-radial measure allowing for the asymmetric treatment of inputs. Towards this end, the disequilibrium model of Maddala and Nelson (1974) is extended so as to allow for the possibility of regime dependent inefficiency. Although the stochastic frontier model that is presented below can be applied to any production process exhibiting nonsubstitution of inputs, the exposition that follows will take the von Liebig model from the crop response literature as the reference framework. The proposed stochastic frontier von Liebig model is then applied to the well known experimental dataset of Heady and Pesek (1954). It should be emphasized at this time that the point of the present study is not to investigate whether the von Liebig function gives a better representation of crop response but to take the model as a starting point and propose a way to incorporate technical efficiency measurement.

Section II discusses the von Liebig hypothesis and the switching model approach already present in the literature, section III introduces Russell's non-radial input-oriented technical efficiency index in the context of the von Liebig crop response model and extends the switching regression suggested by Paris (1992); section IV presents the estimation results and section V provides some concluding remarks and suggestions for future extensions.

## II. The von Liebig Crop Response Model

*Historical Perspective*

Much of the debate surrounding crop response models has been centered around which functional form provides a better representation of crop response to different fertilizer nutrients' levels. Polynomial functions such as a quadratic or square root form which allow for input substitution have been historically popular, since they are relatively easy to estimate. However, the technical restrictions imposed by polynomial specifications has brought about their criticism on the grounds that they force input substitution, do not allow for plateau growth and often over-estimate the optimal fertilizer quantity (Ackello-Ogutu, Paris and Williams, 1985). Following the important contributions in this topic by Paris and his co-authors (Lanzer and Paris, 1981; Grimm, Paris and Williams, 1987; Paris and Knapp, 1989; Paris, 1992; Holloway and Paris, 2002), agronomists and agricultural economists have turned their attention to the von Liebig model as an alternative representation of crop response models.[1] The von Liebig technology reflects the "*law of the minimum*" whereas plant growth is constrained by the level of the scarcest nutrient, exhibiting therefore zero elasticity of factor substitution.

According to Ploeg, Böhm and Kirkham (1999) the main ingredients of the "*law of the minimum*" first appeared in the works of the agricultural chemist Carl Sprengel as reflected by the statement below, appearing in his article of 1828, "… *when a plant needs 12 substances to develop, it will not grow if any one of these is missing, and it will always grow poorly, when one of these is not available in a sufficiently large amount as required by the nature of the plant*". In spite of this, the "*law of the minimum*" has been mostly associated with the name of von Liebig who was a forceful defender of the law (von Liebig, 1855).

The law of minimum of the limiting factors posits two crucial characteristics: first, a yield plateau where plant reaches it's maximum growth and; second, non-substitution between nutrients. The non-substitution characteristic indicates that successively increasing the level of the non-limiting nutrients does not affect the yield, as is the case for a Leontief production function. An implication of this property

[1] However, it should be noted that the first attempts to empirically validate the law of minimum were carried out by Webb (1972) and Waggoner and Norvell (1979) who were the first to develop mathematical techniques for this nineteenth-century law.

is that the isoquants of the crop production function have vertical and horizontal legs that join at right angles as expressed by the formulation below:

$$y_i = min\left\{f_1\left(z_{1i}\right), f_2\left(z_{2i}\right),\ \ldots, f_l\left(z_{ki}\right)\right\} \tag{1}$$

where $y_i$ is the actual level of crop production of the $i^{th}$ individual $\left(i=1,\ 2,\ \ldots,n\right)$, $f_k : \Re_+ \rightarrow \Re_+$ is an arbitrary increasing response function of the $k^{th}$ nutrient level given by $z_{1i},\ \ldots, z_{ki}$, and $\Re_+$ denotes the nonnegative real numbers.[2] The minimum operator selects the level of crop yield that is associated with the limiting nutrient, as declared by the von Liebig's conjecture. The input requirement set for this nonhomothetic[3] technological structure, providing all input combinations capable of producing a given output level, is defined by (Chambers and Lichtenberg, 1996):

$$V\left(y\right) \equiv \left\{\mathbf{z} \middle| min\left\{f_k\left(z_k\right)\right\} \geq y,\ k=1,\ \ldots K\right\} \tag{2}$$

where **z** is the vector of nutrients and $V$ satisfies the correspondence $\Re_+ \rightarrow \Re$.[4] An important property of the above nonhomothetic Leontief technological structure is that the degree of returns to scale can be different for each input (Lau and Tamura, 1972). In addition to the production function and the input correspondence set the following two subsets are important: (a) the isoquant and, (b) the technically efficient subset. In the case of the von Liebig crop response model both sets are defined, respectively, as:

$$Isoq\left(y\right) \equiv \left\{\mathbf{z} \middle| \mathbf{z} \in V\left(y\right),\ \forall k,j=1,\ \ldots K,\ k \neq j\ z_k \geq g_k\left(y\right) \wedge z_j = g_j\left(y\right)\right\} \tag{3}$$

and

[2] As Paris (1992) demonstrated, the model in (1) can be easily made consistent with the law of diminishing marginal productivities by choosing each response function to be concave.

[3] It is nonhomothetic because the expansion path is not necessarily a ray through the origin.

[4] According to Chambers (1988) the von Liebig crop response model is a special case of what he calls *Kohli-output* nonjoint or nonlinear Leontief production technology. It is also a member of the CES family of production functions introduced by Arrow *et al.*, (1961)

$$Eff(y) \equiv \{\mathbf{z} | \mathbf{z} \in V(y),\ \forall k{=}1, \ \ldots K,\ z_k = g_k(y)\} \tag{4}$$

where $g_k(y) = f_k^{-1}(y)$ since $f_k$ is an increasing function. Unlikely with well-behaved technologies the efficient subset of the input correspondence is a subset of the isoquant for each output level $y$.[5] Actually, the efficient subset coincides with the right angle point of the Leontief-type technology isoquants.

The potential yield functions, $f_k$, can be expressed by a wide variety of functional forms such as the linear, quadratic, square-root, Mitscherlich-Baule and non-linear specifications.[6] In order to keep things as simpler as possible, in the present paper we focus on the linear case. However, our model can be easily extended to all other functional specification existing in the literature for the von Liebig crop response model.[7] The linear specification gives rise to the linear-response and plateau model (LRP) and by focusing upon two nutrients, namely phosphorus and nitrogen, we restate the von Liebig crop response model in (1) as:

$$y_i = min\{a_P + \beta_P P_i,\ a_N + \beta_N N_i,\ m\} \tag{5}$$

where $y_i$ is corn yield, $P_i$ and $N_i$ are the applied quantities of the corresponding phosphorus and nitrogen nutrients. The intercepts $\alpha_P$ and $\alpha_N$ are the proportional functions of the nutrients (*i.e.*, phosphorous and nitrogen) available in the soil and they take only positive values. The parameters $\beta_P$ and $\beta_N$ show the slope of the corresponding crop yield response function of the two fertilizers. Finally, *m* is the asymptotic plateau or in other words is the maximum corn yield.

After some level of application of the two nutrients, (*i.e.*, $\bar{P}$ and $\bar{N}$) the plant will no longer respond to the extra-applied level of phosphorous and nitrogen. Paris

[5] The variable elasticity of substitution (VES) and weak input disposability functions are also examples of production functions whose isoquants are not contained in their efficient subsets (Färe and Lovell, 1978).

[6] A comparative evaluation of all the alternative functional specifications in the context of the von Liebig crop response model is provided by Frank, Beattie and Embleton (1990) and Paris (1992).

[7] Paris (1992) argues that the potential yield functions can be also non-linear without the damage of misspecifying the direct relation between nutrients and von Liebig yield function.

(2005) expresses the plateau as $m = min\left\{f_K\left(\bar{K}\right),\ \ldots\ , f_L\left(\bar{L}\right)\right\}$ where $\bar{K}$ and $\bar{L}$ are the fixed levels of the other conventional factors of production (*e.g.*, capital, labor) that are held at non limiting levels. So, in the point *m*, the plant reaches the maximum growth (*i.e.*, plateau) by the use of phosphorus and nitrogen. After this point the plant growth depends exclusively on the use of the conventional factors of production that are not included in the model. In other words, conventional factors of production are arbitrarily fixed at levels presumed to be sufficiently high for causing either nitrogen or phosphorous to be limiting factors.

A single-variable-nutrient illustration of the linear response (LRP) crop yield function is presented in figure 1. In this figure the horizontal axis represents phosphorous nutrient and the vertical axis crop yield. According to the hypotheses stated above, the plant obtain the nutrient (*i.e.*, phosphorous) from two sources: (*a*) the soil (*S*) represented by the negative segment of the horizontal axis and; (*b*) the applied quantity of phosphate fertilizer (*F*) represented on the positive segment of the horizontal axis. When the phosphorous content of soil is zero and no fertilizer is applied, the crop yield will be zero as well. When the applied fertilizer is zero while the soil contains a positive amount of phosphorous that can be absorbed by the plant, the yield would be positive and is represented by the dotted line between the two vertical axes. If, however, positive amount of fertilizer is applied crop yield increases up to a maximum level or plateau, $y^{max}$. At that point, *i.e.*, $\bar{F}_P$, if the amount of fertilizer is further increased, the crop yield will not be affected unless the use of conventional factors of production is altered.

*The Switching Regression von Liebig Crop Response Model*

In many cases, the inputs are not controllable with certainty and experimental error arises. If, following Paris (1992) reasoning, each potential yield function has a specific experimental error associated to it due for instance to the different implications that each nutrient has for the vegetative, maturity and vegetative stage of a crop, then model (5) can be represented by a disequilibrium model as below:

$$y_{Pi} = \alpha_P + \beta_P P_i + \varepsilon_{Pi} \tag{6}$$

$$y_{Ni} = \alpha_N + \beta_N N_i + \varepsilon_{Ni} \tag{7}$$

$$y_{mi} = m + \varepsilon_{mi} \tag{8}$$

where $y_{Pi}$ denotes the crop yield of phosphorus, $y_{Ni}$ the crop yield of nitrogen and, $y_{mi}$ that of the other conventional factors of production, *i.e.*, the plateau. The terms $P_i$ and $N_i$ are the respective applied quantities of each fertilizer, and $\alpha_P$, $\alpha_N$, $\beta_P$ and $\beta_N$ are the unknown parameters need to be estimate. Given (6)-(8) the actual crop yield for observation *i*, is given by:

$$y_i = min\{y_{Pi}, y_{Ni}, y_{mi}\} \tag{9}$$

Under the assumption that the three error terms $\varepsilon_{Pi}$, $\varepsilon_{Ni}$ and $\varepsilon_{mi}$ are pairwise independent, independent of the regressors and following a normal distribution with mean zero and variances $\sigma_P^2$, $\sigma_N^2$ and $\sigma_m^2$ respectively, the unconditional density of $y_i$ is given by (Paris, 1992):

$$\begin{aligned} h(y_i) =& \frac{1}{\sigma_P}\phi\left\{\frac{y_i-(\alpha_P+\beta_P P_i)}{\sigma_P}\right\}\left\{1-\Phi\left[\frac{y_i-(\alpha_N+\beta_N N_i)}{\sigma_N}\right]\right\}\left\{1-\Phi\left(\frac{y_i-m}{\sigma_m}\right)\right\} \\ &+\frac{1}{\sigma_N}\phi\left\{\frac{y_i-(\alpha_N+\beta_N N_i)}{\sigma_N}\right\}\left\{1-\Phi\left[\frac{y_i-(\alpha_P+\beta_P P_i)}{\sigma_P}\right]\right\}\left\{1-\Phi\left(\frac{y_i-m}{\sigma_m}\right)\right\} \\ &+\frac{1}{\sigma_m}\phi\left\{\frac{y_i-m}{\sigma_m}\right\}\left\{1-\Phi\left[\frac{y_i-(\alpha_P+\beta_P P_i)}{\sigma_P}\right]\right\}\left\{1-\Phi\left[\frac{y_i-(\alpha_N+\beta_N N_i)}{\sigma_N}\right]\right\} \end{aligned} \tag{10}$$

which has an interpretation as a mixture of normals and is the basis for maximum likelihood estimation of model given by (6)-(8). In addition to Paris (1992), this model has been also analyzed by Berck and Helfand (1990) but with a different interpretation for the error terms.[8]

## III. A Stochastic von Liebig Crop Response Model

[8] According to Berck and Helfand (1990) the levels of the inputs will vary across individual plots within a field due to nonuniformity in the existing levels of the nutrients or unevenness in the distribution of inputs (p. 986).

*Theoretical Foundations*

According to conventional wisdom, differences in management abilities of different firms can explain why some firms might be less efficient than a benchmark one. In the present setting though where experimental data are used and the levels of inputs have been set according to an experimental setting, inefficiency can be intrinsic to the nature of the experiment. For instance, if 0.4 units of nitrogen are used in 10 identical plots combined with 10 different levels of phosphorus, then application of the *law of the minimum* will reveal that for some plot nitrogen becomes limiting and therefore further increasing phosphorus will yield no further increases in yield. Given this experimental setting we would expect that unless the plateau is reached, then inefficiency can be present in one of the nutrients only. Given this observation, the question still remains as to what is the best way to measure efficiency.

Farrell's seminal work (1957) on productive efficiency has been the basis for most empirical work on technical efficiency measurement. As shown by Färe and Lovell (1978), Farrell's radial notion of technical efficiency leads to two different interpretations. The first Farrell measure is referred as output-oriented technical inefficiency and is modeled as a factor which scales output given input mix and technological conditions. The second, called input-oriented technical inefficiency is specified as a factor that scales input usage given output produced and also technological conditions.[9] In the case of a production technology subject to the *law of the minimum* Farrell's measure could well classify an inefficient input bundle as being efficient since it's a radial measure that constraints the input contraction to be the same across inputs. In contrast, Russell's non-radial input-oriented measure of technical efficiency that allows for different inputs to display different reduction levels is suitable for technologies that exhibit non-substitution among factors of production.

Figure 2 illustrates the above ideas with two inputs (*i.e.*, nitrogen and phosphorous) and a production function given by (5). In figure 2 the individual is producing a given level of output $\left(\bar{y}\right)$ using an input combination defined by point *A*, with $P_1$ units of phosphorous and $N_1$ units of nitrogen. The same level of output can

[9] Färe and Lovell (1978) have shown that these two measures of technical inefficiency coincides *iff* the underlying technology is homogeneous of degree one in inputs (*i.e.*, constant returns to scale).

be produced by reducing the use of both inputs until point B that lies on the isoquant associated with the minimum level of inputs required to produce $\overline{y}$.[10] According to Farrell's definition a radial measure of input-oriented technical inefficiency is defined by the ratio $0C/0A$. In this case both input contractions are the same, *i.e.*, $0P_2/0P_1 = 0N^*/0N_1$. However, point *C* is not the minimum level of inputs required to produce $\overline{y}$, as still phosphorous is used in excessive quantities. This point is on the isoquant $\overline{y}$ but it does not lie in the efficient set of inputs, therefore the technical inefficiency is due to the excess use of the phosphorous input. If we decrease its use until point *B* and leave constant the nitrogen input we produce the same output $\overline{y}$.

Holloway and Paris (2002) where the first who attempted to estimate a frontier von Liebig crop response model. In their empirical study of the effects of water and nitrogen on crop yield using Hexem and Heady (1978) data, they consider a model where both experimental error and the inefficiency term enter additively outside the minimum operator as follows:

$$y_{ik} = min\left(\alpha_w W_{ik},\ \alpha_N N_{ik},\ m\right) + u_{ik} + \varepsilon_i \qquad (11)$$

where $u_{ik}$ is the inefficiency term, $\varepsilon_i$ is the random disturbance, $i = 1, \ldots, N$ denotes the observations and $k = 1, \ldots, 5$ are the five sub-samples they considered.[11] However, a model like that assumes that experimental error is not regime specific and indeed is not a switching regression model. In addition, the resulting efficiency measure which is output-oriented in Farrell's sense, does not allow for individual specific differences in efficiency levels,[12] it precludes the possibility of only one input

[10] It should be noted that in these instances, where the production technology exhibits *L*-shaped isoquants, technical efficiency coincides with productive efficiency as defined by Farrell (1957) since allocative efficiency is always maintained (*i.e.*, the cost-minimized input bundle is always on the left angles of the isoquants). However, this presumes that any change in factor prices does not affect the fixed proportion in which inputs are combined in the production process.

[11] The Hexem and Heady (1978) dataset refers to corn production at five different locations in the US namely, Fort Collins CO, Mesa AR (in 1970 and 1971), Yuma Mesa AR and Yuma Valley AR.

[12] Actually, Holloway and Paris (2002) produced technical efficiency estimates only for each of the *k* subsamples they considered.

being inefficient and more importantly, would not classify a point like $C$ in figure 2 as being technically inefficient.

Instead, Russell's non-radial index can appropriately measure technical inefficiency in the von Liebig crop response model. Using the input correspondence defined in (2), the Russell non-radial technical inefficiency index can be defined as:

$$TE^R(y,\mathbf{z}) \equiv \min_{\theta} \left\{ \frac{\sum_k \theta_k}{\sum_k \vartheta(z_k)} \middle| (\theta_1 z_1, \ \ldots, \ \theta_k z_k) \in V(y) \wedge \theta_k \in [0,1] \forall k \right\} \quad (12)$$

where $\vartheta(z_k)=1$ if $z_k>0$ and $\vartheta(z_k)=0$ if $z_k=0$.[13] Actually the above index is the ratio of two distances computed along rays that diverge. The Russell measure clearly generalizes the Farrell input-oriented measure of technical efficiency, with the latter being the special case of $\theta_k = \overline{\theta} \ \forall k$. In the case of the linear crop response model given in (5), Russell's technical inefficiency index is presented also in figure 2. In order to reach the technical efficient input mix, nutrients (*i.e.*, $N$ and $P$) need to be contracted at different proportions. Specifically, reaching the efficient point $B$ when the input mix is given by $A$ requires bigger input contractions for phosphorus than for nitrogen. In this case Russell's non-radial technical inefficiency is defined as $0D/0A$ which is different than $0C/0A$. Nitrogen need to be reduced by $0N^*/0N_1$, whereas phosphorus by $0P^*/0P_1$ and $0N^*/0N_1 \neq 0P^*/0P_1$. Given the nature of the underlying production technology of the von Liebig crop response model which is characterized by $L$-shaped isoquants, Russell's measure is actually the simple average of the orthogonal non-radial input-specific technical efficiency indices suggested by Kopp (1981).[14] According to Kopp (1981)measures nitrogen-specific technical efficiency is defined as $0N^*/0N_1$, whereas phosphorous-specific as $0P^*/0P_1$.

*Econometric Modeling*

[13] As shown by Russell (1985; 1987), the technical inefficiency index defined in (12) satisfies commensurability, indication and weak monotonicity properties but not that of homogeneity.

[14] Instead of the simple average, Russell technical efficiency measure can be obtained using an unweighted geometric mean.

In contrast with Holloway and Paris (2002) approach, we extend the model given in section II to allow for inefficiency in the use of one or both nutrients and consider a Russell input-oriented measure of technical efficiency. In order to allow for regime dependent technical inefficiency we introduce two stochastic terms $\theta_{Pi}$ and $\theta_{Ni}$ which capture whether each nutrient is used in a technically efficient way, *i.e.*, whether it is possible or not to obtain the same level of crop yield by reducing one or both nutrients. In this case, the model embedded in equations (6) to (9) becomes:

$$y_i = min\left\{\alpha_P + \beta_P\left(\theta_{Pi}P_i\right) + \varepsilon_{Pi},\ \alpha_N + \beta_N\left(\theta_{Ni}N_i\right) + \varepsilon_{Ni},\ m + \varepsilon_{mi}\right\} \qquad (13)$$

If nutrient $z$ $\left(i.e.,\ z = P, N\right)$ is used technically efficient then $\theta_z = 1$, otherwise it should be $\theta_z < 1$. Furthermore, it is assumed that both efficiency terms take values on the interval $\left(0,1\right]$. Note that the switching nature of the model is complicated now by the fact that there are two stochastic terms in each nutrient response that will govern the probabilities of switching from one response function to the other. In order to derive the unconditional density of $y_i$ we make the following assumptions concerning the stochastic terms in (13):

$$\begin{aligned}
&\varepsilon_{Pi} \sim iid\, N\left(0,\sigma_P^2\right),\ \varepsilon_{Ni} \sim iid\, N\left(0,\sigma_N^2\right),\ \varepsilon_{mi} \sim iid\, N\left(0,\sigma_m^2\right),\\
&\theta_{Pi} = exp\left(-u_{Pi}\right),\ u_{Pi} \sim iid\, N^+\left(0,\tilde{\sigma}_P^2\right),\\
&\theta_{Ni} = exp\left(-u_{Ni}\right),\ u_{Ni} \sim iid\, N^+\left(0,\tilde{\sigma}_N^2\right)
\end{aligned} \qquad (14)$$

where $N^+$ denotes the half-normal distribution. In addition, it is assumed that the five stochastic terms are distributed independently of each other and of the regressors. Given the above assumptions, the unconditional density of the three response yields $y_{Pi}$, $y_{Ni}$ and $y_{mi}$, given by $\tilde{f}_P$, $\tilde{f}_N$ and $\tilde{f}_m$ respectively can be derived from the joint density of the corresponding experimental error and the inefficiency term for each of the two nutrient responses and from the density of the experimental error for the plateau response.

Specifically, the joint density of the random error and technical inefficiency terms for each nutrient response function is given by:

$$h'_{zi}\left(\varepsilon_{zi},\theta_{zi}\right)=\frac{2}{2\pi\sigma_z\tilde{\sigma}_z}exp\left(-\frac{\varepsilon_{zi}^2}{2\sigma_z^2}-\frac{\left(-ln\,\theta_{zi}\right)^2}{2\tilde{\sigma}_z^2}\right)\frac{1}{\theta_{zi}} \tag{15}$$

Using a change of variables, the joint density of $y_{zi}$ and $\theta_{zi}$ is obtained as:

$$h'_{zi}\left(y_{zi},\theta_{zi}\right)=\frac{2}{2\pi\sigma_z\tilde{\sigma}_z}exp\left\{-\frac{1}{2}\left[\left(\frac{y_i-\alpha_{zi}-\beta_z\left(\theta_{zi}z_i\right)}{\sigma_z}\right)^2+\left(\frac{-ln\,\theta_{zi}}{\tilde{\sigma}_z}\right)^2\right]\right\}\frac{1}{\theta_{zi}} \tag{16}$$

Accordingly the marginal density of $y_{zi}$ given by $\tilde{f}_{zi}$ can be obtained as:

$$\begin{aligned}\tilde{f}_{zi}\left(y_{zi}\right)=\frac{2}{2\pi\sigma_z\tilde{\sigma}_z}\int_0^1 exp\left\{-\frac{1}{2}\left[\left(\frac{y_i-\alpha_z-\beta_z\left(\theta_{zi}z_i\right)}{\sigma_z}\right)^2\right.\right.\\ \left.\left.+\left(\frac{-ln\,\theta_{zi}}{\tilde{\sigma}_z}\right)^2\right]\right\}\frac{1}{\theta_{zi}}d\theta_{zi}\end{aligned} \tag{17}$$

and its cumulative distribution function is given by:

$$\begin{aligned}P\left(y_{zi}>y\right)=\frac{2}{2\pi\sigma_z\tilde{\sigma}_z}\int_{y_i}^{\infty}\int_0^1 exp\left\{-\frac{1}{2}\left[\left(\frac{y_i-\alpha_z-\beta_z\left(\theta_{zi}z_i\right)}{\sigma_z}\right)^2\right.\right.\\ \left.\left.+\left(\frac{-ln\,\theta_{zi}}{\tilde{\sigma}_z}\right)^2\right]\right\}\frac{1}{\theta_{zi}}d\theta_{zi}dy_i\end{aligned} \tag{18}$$

Since the plateau does not entail any efficiency term the expressions for the marginal density and cumulative distribution of $y_{mi}$ are given by:

$$\tilde{f}_m\left(y_i\right)=\frac{1}{\sigma_m}\phi\left(\frac{y_i-m}{\sigma_m}\right) \tag{19}$$

and

$$P\left(y_{mi} > y_i\right) = 1 - \Phi\left(\frac{y_i - m}{\sigma_m}\right) \tag{20}$$

Under the assumption of the von Liebig crop response model, the observed crop yield can occur when either $y_{Pi} = y_i$ or $y_{Ni} = y_i$ or $y_{mi} = y_i$. Taking into account the pairwise independence of the all stochastic terms, the unconditional density of $y_i$ will be a function of the above marginal densities and can be written as:

$$\begin{aligned} h\left(y_i\right) = & \tilde{f}_p\left(y_i\right) P\left(y_{Ni} > y_i\right) P\left(y_{mi} > y_i\right) \\ & + \tilde{f}_N\left(y_i\right) P\left(y_{Pi} > y_i\right) P\left(y_{mi} > y_i\right) \\ & + \tilde{f}_m\left(y_i\right) P\left(y_{Pi} > y_i\right) P\left(y_{Ni} > y_i\right) \end{aligned} \tag{21}$$

and with $h_z\left(\cdot\right)$ denoting the joint density of $y_{zi}$ and $\theta_{zi}$ we have,

$$\tilde{f}_z\left(y_i\right) = \int_0^1 h_z\left(y_i, \theta_{zi}\right) d\theta_{zi},\ \tilde{f}_m\left(y_i\right) = \frac{1}{\sigma_m} \phi\left(\frac{y_i - m}{\sigma_m}\right) \tag{22}$$

and,

$$\begin{aligned} P\left(y_{zi} > y_i\right) &= \int_{y_i}^{\infty} \int_0^1 h_z\left(y_i, \theta_{zi}\right) d\theta_{zi} dy_i \\ P\left(y_{mi} > y_i\right) &= 1 - \Phi\left(\frac{y_i - m}{\sigma_m}\right) \end{aligned} \tag{23}$$

*Computation of Technical Efficiency*

Computation of nutrient-specific technical efficiency is based on the expression:

$$E\left(\theta_{zi} \middle| y_i\right) = \frac{\int_0^1 \theta_{zi} h_z^a\left(y_i, \theta_{zi}\right) d\theta_{zi}}{h\left(y_i\right)} \quad z = P, N \tag{24}$$

where for each nutrient *z*, $h_z^a\left(\cdot\right)$ denotes the joint density of the inefficiency term and actual crop yield. It should be distinguished from the function $h_z\left(\cdot\right)$ which represents the joint density of the inefficiency terms $\theta_z$ and corresponding response function $y_z$. Then, Russell's non-radial index of input-oriented technical efficiency can be computed from:

$$TE_i^R(y, \mathbf{z}) = \frac{\sum_z E(\theta_{zi} | y_i)}{2} \quad z = P, N \tag{25}$$

The expression in the denominator of (24) can be computed directly from the resulting likelihood function while for the numerator numerical integration is required. In the case of phosphorous, the joint density of actual crop yield and the phosphorous specific inefficiency term, $\theta_{Pi}$, will depend on which of the three regimes, phosphorous limited, nitrogen limited or plateau is binding. Also note that $\theta_{Pi}$ enters only the phosphorous response and is independent of all other stochastic terms. Therefore, $h_P^a(y_i, \theta_{Pi})$ will be the sum of three terms as follows,

$$\begin{aligned} h_P^a(y_i, \theta_{Pi}) &= h_P(y_i, \theta_{Pi}) P(y_{Ni} > y_i) P(y_{mi} > y_i) \quad \textit{phosphorous limiting} \\ &+ \left( \int_{y_i}^{\infty} h_P(y, \theta_{Pi}) dy \right) \tilde{f}_N(y_i) P(y_{mi} > y_i) \quad \textit{nitrogen limiting} \\ &+ \left( \int_{y_i}^{\infty} h_P(y, \theta_{Pi}) dy \right) \tilde{f}_m(y_i) P(y_{Ni} > y_i) \quad \textit{plateau limiting} \end{aligned} \tag{26}$$

where all the above expressions are defined in relations (17) and (18) in the previous section.

Similarly for nitrogen we have:

$$\begin{aligned} h_N^a(y_i, \theta_{Ni}) &= \tilde{f}_P(y_i) \left\{ \int_{y_i}^{\infty} h_N(y, \theta_{Ni}) dy \right\} P(y_{mi} > y_i) \quad \textit{phosphorus limiting} \\ &+ P(y_{Pi} > y_i) h_N(y_i, \theta_{Ni}) P(y_{mi} > y_i) \quad \textit{nitrogen limiting} \\ &+ P(y_{Pi} > y_i) \left\{ \int_{y_i}^{\infty} h_N(y, \theta_{Ni}) dy \right\} \tilde{f}_m(y_i) \quad \textit{plateau limiting} \end{aligned} \tag{27}$$

The above input-oriented measures of technical efficiency allow for a given observation to exhibit different degrees of inefficiency for both nutrients and it could be very well the case inefficiency could occur for one of the nutrients only.

## IV. Estimation Results of the Stochastic von Liebig Crop Response Model

The well known experimental data of Heady and Pesek (1954) were used to estimate the suggested stochastic von Liebig crop response model and the subsequent measurement of non-radial input-oriented technical inefficiency. These data consist of 114 observations relating corn yield response to the application of two nutrients, namely, phosphorus and nitrogen on a calcareous Ida silt loam soil in western Iowa. The agronomic experiments that generated these data utilized an incomplete factoral design as phosphorous and nitrogen were applied in various combinations at different levels. Nine levels of nitrogen and phosphorous were selected with two replications for all 114 plots. The crop population was chosen at 18,000 plants per acre. The range of nitrogen and phosphorous treatments varies from 0 to 320 pounds per acre for the purpose of detecting a phase yield decline if it exists.[15]

The ML estimation results[16] of both the average switching regression and stochastic von Liebig crop response models are presented in table 1. Concerning the average von Liebig switching regression model presented in column 2, all parameters including the variances of the error terms are statistically significant at the 5 per cent level. The estimated growth plateau was found to be 126.81 bushels while, corn yield seems to be equal sensitive to both nitrogen and phosphorous application as the estimated coefficients in both response functions exhibit very close estimates. Our estimates are very close estimates with those reported by other authors who used the same experimental data set (*i.e.*, Frank, Beattie and Embleton, 1990; Paris 1992; Paris, 2005). The differences observed in standard errors are probably due to different computation methods.

Estimates of the stochastic von Liebig crop response model are presented in the third column of table 1. All the estimated parameters are statistically significant at the 5 per cent level, except of the variance of the technical inefficiency term for nitrogen. The later implies that only phosphorous displays a significant degree of technical inefficiency. Examining further this finding, the existence of technical inefficiency in both response function was statistically tested. Conducting a statistical test of no

[15] Paris (2005) estimating a linear von Liebig crop response model utilizing the same data set found no evidence of decline in crop yield. Hence, our empirical model does not account of declining yields.

[16] The maximum likelihood estimation was carried out using Gauss version 3.2.23 computer software.

inefficiency in the use of any nutrient, *i.e.*, $\tilde{\sigma}_z = 0$ is complicated by the fact that under the null hypothesis the parameter of interest lies on the boundary. However, Self and Liang (1987) show that under such circumstances the distribution of the likelihood ratio statistic (LR)[17] follows a mixture of a degenerate $\chi^2(0)$ and $\chi^2(1)$ with weights $1/2$. Using the LR test statistic we fail to reject the hypothesis that the variance of the nitrogen inefficiency term is zero, while on the other hand we didn't fail to do the same for the variance of the phosphorous inefficiency term. Therefore, phosphorus is the only nutrient displaying technical inefficiency which is rather natural given the experimental design under which dataset was generated. Hence, the radial measurement of technical inefficiency would have resulted in biased estimates as it may realize both inputs used in excessive quantities.

Accordingly, the stochastic von Liebig crop response model was re-estimated including a technical inefficiency term only in the phosphorus response function and the results are displayed in the fourth column of table 1. As it can be seen, parameter estimates are almost identical to the ones presented in column 3 where both inefficiency terms are included. This is also true for the value of the log-likelihood functions. Comparing, however, these results with the average switching regime von Liebig crop response model we can see that although the estimated growth plateau remains the same across models there are significant differences in estimates for the constant and slope parameters of the phosphorus response function and the standard deviation of $\varepsilon_P$.

Specifically, the slope of the phosphorous response function increased from 0.9395 in the average switching regime von Liebig crop response model to 1.5266 in the stochastic von Liebig crop response models. Equally, the variance of the random term and the constant parameter in the phosphorous response function has been decreased from 0.2359 and 0.2641 to 0.1333 and 0.2201, respectively, when the model is estimated under technical inefficiency. This implies that the average switching regime von Liebig model provides inadequate estimates of the crop yield response function for phosphorous. Specifically it overestimates the proportional

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[17] The likelihood-ratio statistic is computed as, $\lambda = -2\{ln\, L(H_0) - ln\, L(H_1)\}$, where $L(H_0)$ and $L(H_1)$ denote the values of the likelihood function under the null ($H_0$) and the alternative ($H_1$) hypothesis, respectively.

function of the phosphorous available in the soil and the variability of the experimental error associated with the phosphorous regime, while on the other and, it underestimates the impact of phosphorous content in fertilizers on crop yield. Actually, this implies that the stochastic von Liebig crop response model is not a neutral shift of the average switching regime model. Hence, apart of using excessive quantities of phosphorous nutrient, if technical efficiency is assumed the approximation of the corn production technology would be a biased.

Estimates of input-oriented technical efficiency for phosphorous using relation (24) in the form of a frequency distribution within a decile range are presented in Table 2. Given that only phosphorous seems to be used technically inefficient, these estimates coincides with Russell's non-radial index of input-oriented technical inefficiency defined in (12) (*i.e.*, the numerator has only one element, while the denominator equals to one). On the average, the results indicate that the particular experimental design has not been successful in implementing the best practice corn production technology and achieving the maximum possible output of nutrient application. Mean input-oriented technical efficiency was 72.9 per cent implying that experimental plots could have produced the observed corn quantity using on the average about 27 per cent less of phosphorous quantity with the current state of technology and nitrogen application. Moreover, technical inefficiency scores vary considerably across plots, ranging from a minimum of 34.5 per cent to a maximum of 98.7 per cent. The majority of the plots belong to the 70-80 per cent interval. Of the 114 plots in the sample only 11 (*i.e.*, less than the 10 per cent of the plots examined) achieved input-oriented technical inefficiency in phosphorous application above 80 per cent.

## V. Concluding Remarks

When the underlying production process is characterized by a Leontief-type technology, *i.e.*, zero elasticity of substitution, the traditional Farrell (1957) radial index of technical inefficiency is not appropriate. In particular it is possible to identify a decision making unit as being technically efficient although this may not be true. This is because in Leontief-type technologies with *L*-shaped isoquants, the efficient subset coincides with the right angles where the horizontal and vertical legs are joined. Hence, an observation lying either on the vertical or the horizontal leg of the isoquant may be identified with Farrell's radial index as being technical efficient although this is not true. In these instances, Russell's non-radial index of technical efficiency is more appropriate as it allows for inefficiency to be displayed by a subset of inputs only.

Along these lines, the present paper presents an econometric model that can be used to uncover inefficiencies in such circumstances by combining the traditional stochastic frontier framework with switching regression models. For the exposition of our assertion we use the von Liebig model from the well documented crop response literature as the reference framework. Specifically, we extent Paris (1992) average switching regression von Liebig crop response model providing a theoretical consistent framework for the quantitative measurement of non-radial input-oriented technical efficiency in nutrient application. The model has been applied to the well known experimental agronomic data of Heady and Pesek (1954) where yield is related to two nutrients namely, phosphorous and nitrogen. We assume a linear crop response crop yield although our model can be extended to all other specifications suggested from the relevant literature. The empirical results suggest that only phosphorous nutrient has been used in excessive quantities whereas nitrogen application was technically efficient. The nature of the experimental data which was generated utilizing an incomplete factoral design may be behind that finding.

Finally, the present estimation framework can be equally applied to other production technologies characterized by nonsubstitution like petrochemical industries (Lau and Tamura, 1972), steam power plants (Komiya, 1962), paper plants (Ozaki, 1969), iron and steel industry (Nakamura, 1990), bus transportation (Holvad *et al.*, 2004) etc. Further, one interesting extension would be to consider other distributional assumptions for the inefficiency term.

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**Figure 1:** Linear Response Representation of Crop Yield to Phosphorus.

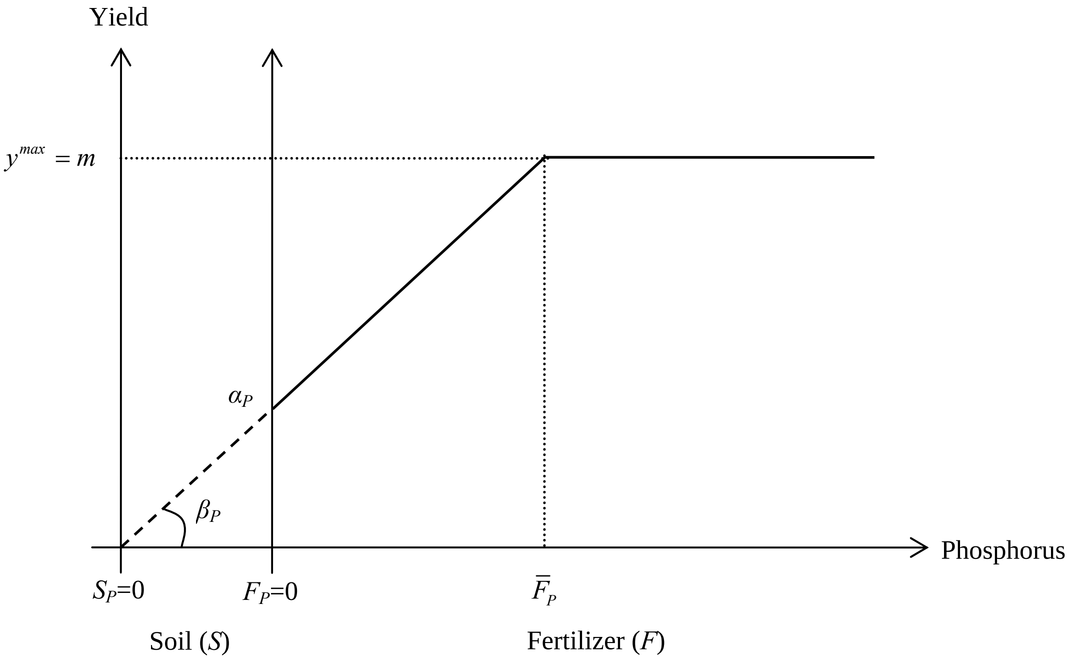

**Figure 2:** Farrell's and Russell's Measures of Input-Oriented Technical Inefficiency Under von Liebig Crop Response Model.

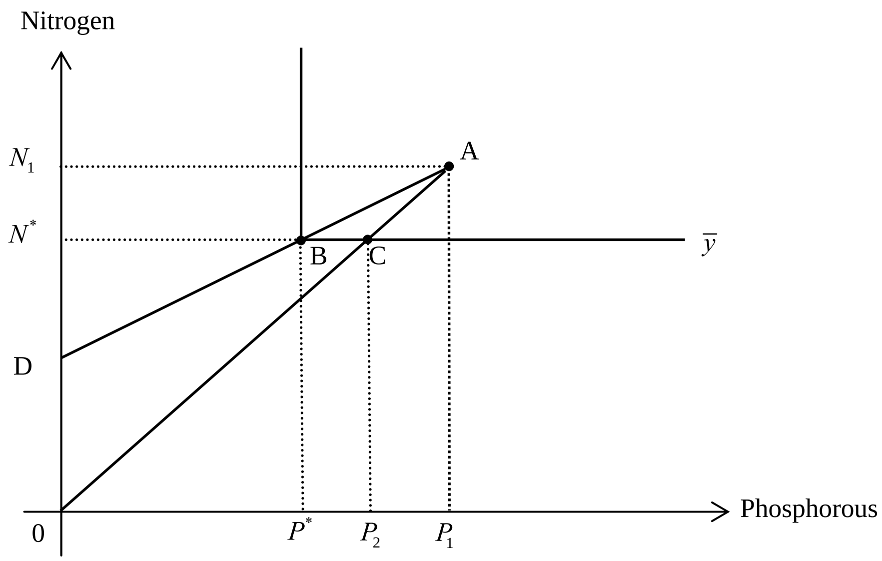

**Table 1.** ML Estimates of the Average and Stochastic von Liebig Crop Response Models.

| Parameter | Average von Liebig Model | | Frontier von Liebig Model | | Restricted Frontier von Liebig Model $\left(\tilde{\sigma}_N = 0\right)$ | |
|---|---|---|---|---|---|---|
| $\alpha_p$ | 0.2641 | (0.0534) | 0.2201 | (0.0302) | 0.2201 | (0.0302) |
| $\alpha_N$ | 0.2895 | (0.0307) | 0.2909 | (0.0306) | 0.2910 | (0.0307) |
| $\beta_p$ | 0.9395 | (0.1016) | 1.5266 | (0.0894) | 1.5266 | (0.0894) |
| $\beta_N$ | 0.9824 | (0.0697) | 0.9875 | (0.0724) | 0.9785 | (0.0693) |
| $m$ | 1.2681 | (0.0144) | 1.2681 | (0.0144) | 1.2681 | (0.0144) |
| $\sigma_P$ | 0.2359 | (0.0304) | 0.1333 | (0.0248) | 0.1333 | (0.0248) |
| $\sigma_N$ | 0.1294 | (0.0168) | 0.1293 | (0.0168) | 0.1294 | (0.0168) |
| $\sigma_m$ | 0.0935 | (0.0104) | 0.0926 | (0.0106) | 0.0926 | (0.0106) |
| $\tilde{\sigma}_P$ | - | - | 0.4406 | (0.0880) | 0.4406 | (0.0880) |
| $\tilde{\sigma}_N$ | - | - | 0.0114 | (0.0261) | - | - |
| *LogL* | 70.651 | | 77.006 | | 77.004 | |
| $N$ | 114 | | 114 | | 114 | |

**Table 2.** Frequency Distribution of Russell's Input-Oriented Technical Efficiency Levels for Phosphorus Nutrient.

| Technical Efficiency (%) | No of Observations |
|---|---|
| 0-10 | 0 |
| 10-20 | 0 |
| 20-30 | 0 |
| 30-40 | 1 |
| 40-50 | 2 |
| 50-60 | 2 |
| 60-70 | 1 |
| 70-80 | 97 |
| 80-90 | 10 |
| 90-100 | 1 |
| Mean | 0.729 |
| Median | 0.727 |
| Minimum | |
| Maximum | |