Control Bands for Tracking Constant Portfolio Allocations with Fixed and Proportional Transaction Costs

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ABSTRACT *The vast majority of research related to optimal asset allocation strategies in the presence of transaction costs, requires formulation of highly sophisticated numerical schemes for the estimation of no-transaction bands; moreover, the optimization objectives examined are far less compared to the number of works that assume frictionless trading. In this article, we point out that an investor may alternatively try to track a constant allocation strategy as derived under the frictionless markets hypothesis and any optimization objective, by applying a loss function that reflects his/her risk preferences. We focus in the two-asset case (one riskless and one risky) and assume a fixed cost per transaction plus a cost proportional to the change in the risky fraction process. Using a recently proposed transformation of the risky fraction process by Nagai (2005), we derive optimal rebalancing policies for the quadratic loss case, using two alternative methods. First, we calculate no transaction bands for investors who choose the boundaries of the bands and their optimal rebalancing actions so that they minimize long run cost per unit time. The latter is defined as the expected cost per transaction cycle (opportunity cost/tracking error plus transaction cost) divided by the expected cycle time. In the second case, the objective is to minimize the expected discounted squared tracking error plus discounted transaction costs over an infinite horizon. On that purpose, similar to Suzuki and Pliska (2004), we use impulse control theory in a continuous-time, dynamic setting and characterize the optimal strategy in terms of a quasi-variational inequality. For both formulations, we derive explicit solutions, which we use to perform sensitivity analysis for the control bands with respect to the market parameters and the magnitude of the transaction costs.*

KEY WORDS: risky fraction process; stochastic impulse controls; control bands; quasi-variational inequalities.

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Introduction

Since R. Merton's (1971) pioneering work on optimal consumption/investment decisions for investors that may place proportions of their wealth in risky assets ("stocks") whose prices are described by geometric Brownian motions and a bank account (or "bond") paying a fixed interest rate under the objective of maximizing lifetime HARA utility of consumption, several articles have emerged in the literature using the same market specifications but different objective functions. To name a few, Pliska (1986) derived optimal strategies for investors that aim to maximize exponential utility of terminal wealth, Browne (1999, 2000) related the probability of achieving a given target performance to the time it takes to achieve it and Young (2004) presented strategies that minimize the probability of lifetime ruin. Depending on the optimization objective, the optimal asset allocation may be constant as in Merton (1971) and Browne (2000), time dependent as in Pliska (1986), or state (wealth) dependent as in Browne (1999) or Young (2004); in the risky wealth-risk free wealth space the former result postulates that portfolio holdings should be located on the so-called Merton line. In the aforementioned models, information arrives continuously and, since investors trade costlessly, optimal policies entail continuous trading; hence following such strategies in the presence of transaction costs will lead to immediate ruin.

Since the early nineties, there has been a wave of research focusing on the removal of one of the most significant simplifying assumptions of Merton's model: frictionless trading. Efforts for the removal of the frictionless market hypothesis date back to the path-breaking work of Davis and Norman (1990) who assumed a cost proportional to the size of each transaction for investors that may invest in a single risky and a riskless asset, aiming to maximize lifetime HARA utility of consumption. Thereafter various articles have appeared studying the optimal transaction policy for an agent facing proportional transaction costs in the financial markets. Shreve and Soner (1994) refined the work of Davis and Norman using viscosity solutions, Dumas and Luciano (1991) studied the problem of maximizing HARA utility of terminal wealth in the limit as the horizon gets very large, and later, Gennotte and Jung (1994) and Liu and Lowenstein (2002) also focused on maximizing HARA utility of terminal wealth. More recently, Nazareth (2002), solved the problem as formulated in Davis and Norman, assuming that the constant of proportionality for the transaction costs is random. Demchuk (2002) assumed that transaction costs are represented by a concave function of the size of the trade in the risky asset and solved the problem under a framework similar to that presented in Gennotte and Jung. Hence, the literature on proportional transaction cost models has focused on HARA utilities; the only exception till now is Weerasinghe (1998) who presented optimal strategies for investors aiming to maximize the probability of reaching a wealth level before going bankrupt. Optimal control actions in markets that contain one risky and one riskless asset and rebalancing entails proportional transaction costs, are of a "local time" type, i.e. the fundamental process can move freely inside a prescribed region. If it reaches its boundary, the controller will simply hold it inside the region by performing the minimal action to avoid crossing of the boundary. Hence, policies are totally specified by the boundaries of the "non-intervention" region. The value function of the optimization problem is typically characterized as a solution of a variational inequality.

Morton and Pliska (1995), observed that optimal trading strategies derived by models that incorporate transaction costs proportional to the amount traded, are not of finite variation; thus these strategies still consist of making infinitesimally small transactions which is not the case in real world. They assumed transaction costs proportional to wealth and derived control bands for investors aiming to maximize their long-run growth rate of wealth. The optimal policy in their model is of finite variation: each time the bond-stock proportion hits the boundaries of the no-transaction band the investor brings it back to an optimal level within the band. Bielecki and Pliska (2000) and Bielecki *et al*. (2004), extended the aforementioned model to risk sensitized growth-rate optimizing criteria. Korn (1998, 1999), added a fixed cost part to the transaction costs part that is proportional to the amount traded in the risky asset. Using the impulse control method, he obtained optimal strategies that also consist of finitely many actions on finite time intervals; he solved the impulse control problem for an investor who maximizes his/her exponential utility of terminal wealth. In this case, the presence of the fixed cost component forces the controller to move the underlying process away from the boundary of the "non-intervention" region. Here, the value function is characterized as a solution of quasi-variational inequalities (qvi). Later, Oksendal and Sulem (2002), based on the theory of viscosity solutions applied to quasi-variational inequalities, presented a numerical scheme that optimizes lifetime HARA utility of consumption. Zakamouline (2002, 2004) presented numerical schemes for investors aiming to maximize constant absolute risk aversion (CARA) utility of terminal wealth and Liu (2004) derived optimal impulse control bands for investors seeking to optimize lifetime CARA utility of consumption. The reader should note that most research work in transaction costs models is done for simple two-asset markets; due to computational intractability, there is only a handful of articles that consider (correlated) multi-asset markets; see Akian et al. (1996, 2001), Atkinson and Mokkhavesa (2004), and Muthuraman (2006 a,b,c). For some survey articles, the interested reader may also consult Cadenillas (2000) and Zariphopoulou (1999).

In a recent article, Korn (2004) highlighted the difficulties related to the application of transaction cost models in real world tasks. The vast majority of methods require formulation of highly sophisticated numerical schemes for the derivation of optimal allocation rules; thus, it is difficult for a practitioner to derive optimal control bands within which his/her bond-stock proportions should lie and rebalancing points to which the proportions should be driven when they hit the boundaries of the bands. Moreover, the optimization objectives examined are far less compared to models that adopt the frictionless markets hypothesis; for instance probability-related objectives as in Browne (1999, 2000), Young (2004) and Bayraktar and Young (2004), have not been examined yet. In this work, we point out that to enhance tractability, an investor may alternatively try to track a constant optimal portfolio strategy as derived under the frictionless markets hypothesis and any optimization objective, using an appropriate loss function that reflects his/her risk aversion. Based on a transformation of the risky fraction process recently proposed by Nagai (2005) and inspired by a simple cash inventory model presented in Karlin and Taylor (1981, section 15.4), we present a simple method for the derivation of optimal rebalancing rules for investors that aim to minimize their long run tracking error plus transaction cost, per unit time. One advantage over the well-developed method in which one minimizes discounted tracking error plus transaction cost over an infinite horizon is that it does not require the transaction cost coefficients to be constant over lifetime; different valid control bands can be calculated for different transaction cost levels. Given that the cost component that is proportional to the amount of the transaction is mostly related to the bid-ask spread, this method suits better to real world applications. Furthermore, we use Nagai's transformation to apply an impulse control model similar to the one presented in Pliska and Suzuki (2004) to derive control policies for investors that aim to minimize discounted lifetime tracking error plus discounted transaction costs. To illustrate our methodology we examine a loss function that penalizes squared deviations from the desired proportions in the original scale and compare the policies implied by the two alternative methods in an extensive application.

Hence, in this article we present models for tracking a specific constant target asset mix (which may have been derived under the frictionless markets hypothesis) in the presence of constant and proportional transaction costs. Models for tracking a constant target asset mix have been proposed by Leland (2000) and Suzuki and Pliska (2004). In both articles researchers aim to minimize discounted *lifetime* tracking error plus transaction costs. Here we also show how one may derive strategies that minimize long run cost (tracking error/opportunity cost plus transaction cost) *per unit time*. Apart from that, our work differs with respect to Leland's in that we use the exact expression for the stochastic process representing the evolution of portfolio proportions in the absence of interventions; moreover, we include a fixed component to the cost of each transaction in addition to Leland's component that is proportional to the change in an asset proportion. The way we treat the "minimization of lifetime discounted tracking error plus discounted transaction costs" problem is similar to Suzuki and Pliska 's treatment with two differences. First, we use Nagai's (2005) transformation on the risky fraction process –apparently, this transformation simplifies substantially the system of nonlinear equations that need to be solved so that control policies are derived. Second, the objective function we consider differs in that it solely penalizes tracking error.

Similar to the impulse control models presented in Korn (1998, 1999), Oksendal and Sulem (2002), Zakamouline (2002, 2004) and Liu (2004), the optimal strategy is characterized by four unknown parameters L, l, u, U. If the proportion of the risky asset hits level L (or U), then a transaction is made so that it resumes at level l (or u). Estimation of the inner (l, u) and outer boundaries (L, U) for both short and long term objectives pertains to the solution of a system of nonlinear equations. Computations can be significantly reduced, for portfolio managers that seek to find just the optimal outer boundaries and rebalance to a predefined optimal allocation. The predefined rebalancing point may be the optimal allocation as derived under the frictionless markets hypothesis for any optimization objective that suits best the portfolio manager. This simplification (adopted also in Korn, 2004), facilitates computations drastically and as shown in the application the obtained results are quite close to the ones obtained from the solution to the full problem.

The plan for our paper is as follows. In the following section, we display our two-asset market model, formulate Nagai's (2005) transformation for the risky fraction process and present a precise statement of the portfolio manager's optimization objective. In the third section, we explain how optimal trading strategies can be computed via standard diffusion theory for investors that minimize long run cost per unit time. The cost is comprised by two components: a transaction cost that is linear in the change of the (transformed) risky fraction process that occurs in every transaction and an opportunity cost/tracking error that is dependent on the investor's risk preferences via an associated loss function. In the fourth section we illustrate how the optimal rebalancing policies, characterized by four parameters $L<l<u<U,$ can be derived by solving a certain quasi-

variational inequality (qvi). In this case, the objective is to minimize the expected discounted tracking error plus discounted transaction costs over an infinite horizon. The fifth section elaborates on the density function of the controlled risky fraction process and the resulting probability density for the optimally controlled wealth. Section 6 contains a specific example and a sensitivity analysis, where we show with numerical examples how the control bands depend upon the values of individual input parameters. We conclude with some final remarks and directions for further research in section 7.

Problem formulation and the risky fraction process

We consider the simple two-asset market model, in which the set of securities consists of one bond, whose price $S^0(t)$ is described by the following ordinary differential equation:

$$dS^0(t) = rS^0(t)dt, \qquad S^0(0) = s^0, \tag{2.1}$$

and one risky asset with price $S^1(t)$ that is governed by the stochastic differential equation:

$$dS^1(t) = S^1(t)(\mu dt + \sigma dW_t), \qquad S^1(0) = s^1 \tag{2.2}$$

where W_t is a standard Wiener process defined on a filtered probability space (Ω, F, P, F_t). We assume that F_t satisfies the usual conditions, namely it is right continuous and F_0 includes all P-null sets in F, and that $\sigma^2 > 0$. Let $(p^0(t), p^1(t))$ be the shareholding process, to be chosen by the portfolio manager, each component of which represents the number of shares for the i-th asset at time t. It is required to be a piecewise constant, adapted process. Denote by $V(t) := \sum_{i=0}^{1} p^i(t)S^i(t)$ the wealth process or value process, which is strictly positive for all $t \geq 0$. Now we may define the risky fraction process $b^i(t)$ by setting

$$b^i(t) = \frac{p^i(t)S^i(t)}{V(t)}, \qquad i = 0,1 \tag{2.3}$$

and for later use we set $b(t)=b^1(t)$. We prohibit short selling and borrowing so for each t we require $b(t) \geq 0$ and $b(t) \leq 1$. Under the condition of self-financing $V(t)$ satisfies

$$dV(t) = V(t)\left(\sum_{i=0}^{1} b^i(t)\frac{dS^i(t)}{S^i(t)}\right) = b^0(t)rdt + b^1(t)(\mu dt + \sigma dW_t), \qquad V(0) = v \tag{2.4}$$

and we have

$$\frac{dV(t)}{V(t)} = (r + b(t)(\mu - r))dt + b(t)\sigma dW_t, \qquad V(0) = v \ . \tag{2.5}$$

The risky fraction process was first studied by Morton and Pliska (1995). Using Ito's formula, they showed that, for the two-asset case, it evolves according to the following stochastic differential equation

$$db_t = b_t(1-b_t)(\mu - r - \sigma^2 b_t)dt + b_t(1-b_t)\sigma dW_t \, . \tag{2.6}$$

To ease calculations in later sections, we adopt the 1-1 transformation proposed recently by Nagai (2005), defined by

$$y = \psi(b) := \log b - \log(1-b) \tag{2.7}$$

and one may easily observe the form of the inverse mapping φ:

$$\phi(y) := \frac{\exp y}{1+\exp y}. \tag{2.8}$$

Using once again Ito's formula, the evolution of y is formulated as a geometric Brownian motion with constant drift

$$dy_t = \kappa dt + \sigma dW_t \tag{2.9}$$

where $\kappa = \mu - r - \frac{\sigma^2}{2}$; a certainly more manageable form compared to (2.6).

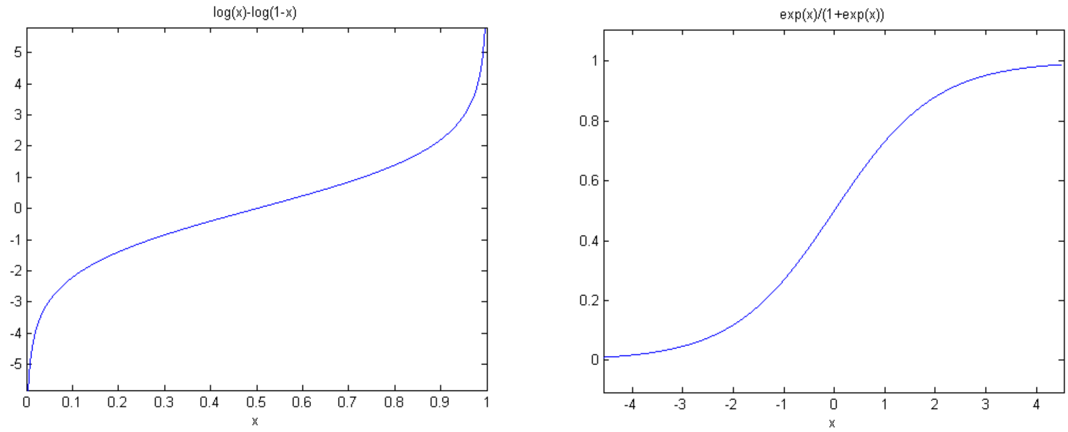

Figure 1. Nagai's transformation and its inverse for the two-dimensional case.

We now turn to the specification of the transaction cost, which is essentially the same as in Suzuki and Pliska (2004). If the transformed risky proportion is y and a transaction is made resulting the new risky proportion $\tilde{y}$, then the transaction cost incurred at that time is

$$c(y,\tilde{y}) := K + k|y-\tilde{y}| \tag{2.10}$$

where K and k are two suitably chosen (so that the scale transformation is accounted for), strictly positive scalars. Thus, the linear component is proportional to the change in transformed proportions and not, as is common in much of the transaction cost literature, proportional to the dollar amount of the transaction. Because of the fixed cost component, it suffices to consider trading strategies of the form $\{(\tau_n, y_n)\}$, where τ_n is the time of the nth transaction and y_n the risky proportion that results from the nth transaction. $\{(\tau_n, y_n)\}$ must satisfy some standard technical requirements: τ_n is a stopping time, $\tau_n < \tau_{n+1}$, $\tau_n \to \infty$ as $n \to \infty$, and y_n is F_{τ_n}-measurable. The advantage for such a specification for transaction costs is that it facilitates computations; a disadvantage is that for practical applications a portfolio manager should recalibrate transaction cost parameters for investors of different wealth levels.

We study two different methods for the derivation of the optimal rebalancing strategies and in the application we compare the control bands they suggest. To illustrate our methodology we use a quadratic function for the deviation of the risky fraction process from the target level in the original scale. The reader should note that a portfolio manager may choose from a large variety of loss functions to represent his/her opportunity costs –the quadratic functions here are chosen for computational simplicity in the application. For example, tracking error may be represented by an exponential function of the deviation of the observed from the desired proportion or a function of

HARA type with a suitably chosen risk aversion parameter. Such loss functions are justified by the findings of Rogers (2001) and Cover (1991) who observed that the payoff of a fixed-proportion rule is quite insensitive to the chosen proportion in a neighborhood of the Merton proportion. The target levels of the risky fraction process in the original scale may for instance be equal to

- $p_1 = \frac{\mu - r}{\sigma^2}$, the risky asset proportion that maximizes log utility and the portfolio's exponential growth rate.
- $p_2 = \frac{\mu - r}{(1-\gamma)\sigma^2}$, $\{\gamma \in \Re : \gamma < 1, \gamma \neq 0\}$ the risky asset proportion that maximizes HARA utility with risk aversion parameter γ.
- $p_3 = fp_1$ $f \in (0,1]$, an efficient fractional Kelly strategy that maximizes capital growth and at the same time achieves a given probability of maintaining an accumulated risk free return (see Li, 1993).
- a target proportion exogenously specified by a portfolio manager or one that follows an index as in Suzuki and Pliska (2004).

In the impulse control method, which is treated in the fourth section, the objective is to minimize the expected discounted squared tracking error plus transaction costs over an infinite planning horizon. Let p and π denote the target proportion of wealth in the risky asset in the original and the transformed scale respectively. Then, under an admissible trading strategy $\{(\tau_n, y_n)\}$ and given an initial proportion vector $b(0)=b_0$, the objective function is given by

$$J(y_0, \{(y_n, b_n)\}) := \mathrm{E}_{y_0}^{\{(\tau_n, y_n)\}} \left[\lambda \int_0^\infty e^{-\beta t} g(y(t)) dt + \sum_{n=1}^{\infty} e^{-\beta \tau_n} c(y(\tau_{n-}), y_n) I_{\{\tau_n < \infty\}} \right], \tag{2.11}$$

where

$$\begin{aligned} & g(y(t)) = \left(e^{y(t)-\pi} - 1\right)^2 \\ & \beta > 0 \qquad , \\ & \pi \in (-\infty, \infty) \end{aligned} \tag{2.12}$$

and $c(y(\tau_{n-}), y_n)$ as in (2.10). In (2.11) the first term measures discounted tracking error/opportunity costs and the second discounted transaction costs; β is a discount factor and λ is a constant chosen by the portfolio manager to reflect his/her loss preferences. The portfolio manager seeks an admissible trading strategy minimizing $J(y_0, \{(\tau_n, b_n)\})$. Hence, one would like to compute the value function

$$J(y_0) := \inf_{\{(\tau_n, y_n)\}} J(y_0, \{(\tau_n, y_n)\}) \tag{2.13}$$

where the infimum is taken, over all admissible trading strategies, and find the trading strategy that attains this infimum.

Minimization of long run cost per unit time

Problem formulation

The usual practice for dealing with tracking problems in the presence of constant and proportional transaction costs where a finite number of actions per finite time intervals is required, is to minimize lifetime discounted tracking error plus discounted transaction costs via solving a system of quasi-variational inequalities (qvi). This approach has been adopted for example in Cadenillas and Zapatero (1999) and in Baccarin (2002) for tracking exogenously specified target levels of an exchange rate and for cash management respectively. In this section, we minimize *long run cost per unit time*. This approach uses simple mathematical tools from diffusion theory; thus, no knowledge of impulse control theory is required. Similar to the qvi approach, estimation of the inner and outer control bands pertains to the solution of a system of nonlinear equations. Unfortunately, these nonlinear equations turn out to be significantly more complex compared to the ones derived from the qvi approach. Nevertheless, one may relatively easily derive them using any software that performs symbolic calculations and solve the resultant system using standard routines that perform algorithms like Newton-Raphson or one of its descendants.

Let y_t be the transformed (by 2.7) proportion of wealth an investor has in the risky asset at time t. In the absence of intervention, y_t behaves as the geometric Brownian motion (2.9). Deviations from the (pre-specified) optimal fraction π, involve an opportunity cost since part of wealth is not optimally invested. We therefore suppose that holding stocks at level y_t for the transformed risky fraction process incurs opportunity costs at a quadratic rate in the original scale

$$g(y_t) = \lambda\left(e^{(y_t-\pi)} - 1\right)^2 . \tag{3.1}$$

Now consider the following control band policy for the transformed risky fraction process: "If the transformed risky fraction process reaches level U above the target level π, reduce its level to u. This transaction incurs a cost of $K + k(U - u)$. If the transformed risky fraction process reaches level L below the target level π, increase its level to l. This transaction incurs a cost of $K + k(l - L)$." Define a cycle to be from one intervention returning the level to l or u from L or U, to the next such intervention; the long-run cost per unit time will be the expected cost per cycle divided by the expected cycle time, or

$$\frac{C + A}{B} \tag{3.2}$$

where C represents the expected transaction cost per cycle, A denotes the expected opportunity cost per cycle, and B stands for the expected cycle time.

Related results from diffusion theory

To derive A, B and C we use standard diffusion theory as in Karlin and Taylor (1981), or Borodin and Salminen (2002). Assume U and L be fixed subject to $-\infty < L < U < \infty$, and define $T(s) = T_s$ be the hitting time of s for the y process. Throughout the paper we let

$$T^* = T_{U,L} = \min\{T(U), T(L)\} = T(U) \wedge T(L) \tag{3.3}$$

be the first time the process reaches U or L. To proceed, we need to be able to calculate the following quantities for the transformed risky fraction process:

$$v_1(y) = \Pr\{T(U) < T(L) | Y(0) = y\} \qquad L < y < U \tag{3.4}$$

the probability the process reaches U before L,

$$v_2(y) = E\left[T^* | Y(0) = y\right] \qquad L < y < U \tag{3.5}$$

the mean time to reach U or L, and

$$v_3(y) = E\left[\int_0^{T^*} g\big(Y(t)\big)\Big|Y(0) = y\right] \qquad L < y < U \tag{3.6}$$

for a bounded and continuous function g. v_1, v_2 and v_3 need to satisfy the following differential equations (Karlin and Taylor, chapter 15):

$$\kappa\frac{dv_1}{dy} + \frac{\sigma^2}{2}\frac{d^2v_1}{dy^2} = 0 \ \text{ for } \ L < y < U\,,\ v_1(L) = 0, \quad v_1(U) = 1\,; \tag{3.7}$$

$$\kappa\frac{dv_2}{dy} + \frac{\sigma^2}{2}\frac{d^2v_2}{dy^2} = -1 \ \text{ for } \ L < y < U\,,\ v_2(L) = v_2(U) = 0\,; \tag{3.8}$$

$$\kappa\frac{dv_3}{dy} + \frac{\sigma^2}{2}\frac{d^2v_3}{dy^2} = -g(y) \ \text{ for } \ L < y < U\,,\ v_3(L) = v_3(U) = 0\,. \tag{3.9}$$

For the solutions of these problems, let

$$S(y) = \int^y s(\eta)d\eta \tag{3.10}$$

denote the scale function of the y_t process where

$$s(y) = \exp\left\{-\int^y \left[2\kappa/\sigma^2\right]d\xi\right\} \tag{3.11}$$

and

$$m(y) = 1/\left[\sigma^2 s(y)\right], \tag{3.12}$$

denote the speed density of the y_t process. The solution to (3.4) is

$$v_1(y) = \frac{S(y) - S(L)}{S(U) - S(L)} \ \text{ for } \ L \le y \le U\,. \tag{3.13}$$

(3.5) is a special case of (3.6) with g equal to the indicator function. The solutions to (3.8), (3.9) are formulated as follows:

$$v_2(y) = 2\left\{v_1(y)\int_y^U \left[S(U) - S(\xi)\right]m(\xi)d\xi + \left[1 - v_1(y)\right]\int_L^y \left[S(\xi) - S(L)\right]m(\xi)d\xi\right\} \tag{3.14}$$

$$v_3(y) = 2\left\{v_1(y)\int_y^U \left[S(U) - S(\xi)\right]m(\xi)g(\xi)d\xi + \left[1 - v_1(y)\right]\int_L^y \left[S(\xi) - S(L)\right]m(\xi)g(\xi)d\xi\right\}. \tag{3.15}$$

Calculation of expected costs per transaction cycle

The scale function for the geometric Brownian motion (2.9) that corresponds to the transformed risky fraction process, is

$$S(y) = \exp\left(-2\kappa y/\sigma^2\right) \tag{3.16}$$

and the speed measure is

$$m(y) = \exp\left(2\kappa y/\sigma^2\right)/\sigma^2\,. \tag{3.17}$$

Using (3.16) the expected transaction cost per cycle is

$$\begin{aligned} C &= K + \Pr\{T(U) > T(L)|Y(0) = u\}k(U-u) + \Pr\{T(U) > T(L)|Y(0) = l\}k(U-u) \\ &\quad + \Pr\{T(L) > T(U)|Y(0) = u\}k(l-L) + \Pr\{T(L) > T(U)|Y(0) = l\}k(l-L) \\ &= K + \left(v_1(u) + v_1(l)\right)k(U-u) + \left(2 - v_1(u) - v_1(l)\right)k(l-L) \end{aligned} \tag{3.18}$$

with $v_1(.)$ given by (3.13). In words, the expected transaction cost per transaction cycle is comprised by five components: a constant, two parts proportional to the difference between the upper boundary and the upper rebalancing point weighted by the probabilities of reaching the upper boundary from the upper and lower rebalancing points and two parts proportional to the difference between the lower boundary and the lower rebalancing point weighted by the probabilities of reaching the lower boundary from the upper and lower rebalancing points.

For the expected cycle time, using (3.14), (3.16), (3.17), we obtain

$$B = \left(v_1(u) + v_1(l)\right)v_2(u) + \left(2 - v_1(u) - v_1(l)\right)v_2(l) \tag{3.19}$$

where

$$\begin{aligned} v_2(y) &= 2\frac{U\left(S(y) - S(L)\right) + y\left(S(L) - S(U)\right) + L\left(S(U) - S(y)\right)}{\left(S(L) - S(U)\right)\sigma^2} \\ &= \frac{2y}{\sigma^2} - \frac{2U}{\sigma^2}v_1(y) - \frac{2L}{\sigma^2}\left(1 - v_1(y)\right) \end{aligned}. \tag{3.20}$$

Similarly, the expected opportunity cost/tracking error per transaction cycle is

$$A = \left(v_1(u) + v_1(l)\right)v_3(u) + \left(2 - v_1(u) - v_1(l)\right)v_3(l) \tag{3.21}$$

where

$$v_3(y) = -v_1(y)h(U) - \left(1 - v_1(y)\right)h(L) \tag{3.22}$$

and

$$\begin{aligned} h(y) = {} & \frac{\lambda\left(\left(4\kappa^3 + 6\sigma^2\kappa^2 + 2\sigma^4\kappa\right)\left(y - \pi\right) + 7\sigma^2\kappa^2 - \sigma^6 + \left(2\kappa^3 + \sigma^2\kappa^2\right)e^{(2(y-\pi))}\right)}{\sigma^2\kappa\left(\sigma^2 + 2\kappa\right)\left(\sigma^2 + \kappa\right)} \\ & - \frac{\lambda\left(8\left(\kappa^3 + \sigma^2\kappa^2\right)e^{(y-\pi)} + 6\kappa^3 + \sigma^6 S(y - \pi)\right)}{\sigma^2\kappa\left(\sigma^2 + 2\kappa\right)\left(\sigma^2 + \kappa\right)} \end{aligned}. \tag{3.23}$$

Hence, by using (3.18)-(3.23) we derived the quantity $\frac{C + A}{B}$. To minimize with respect to *L, l, u* and *U*, one should take the corresponding derivatives and equate them to zero. Since these expressions are lengthy, we omit them for space economy[1]. The derivatives form a system of nonlinear equations, which can be solved computationally using the Newton-Raphson algorithm or one of its successors; numerical results are presented at the sixth section.

Some remarks are worth considering. First, instead of seeking two optimal rebalancing points (an inner band), one may simplify the problem by considering a single rebalancing point where the process is driven when it reaches *L* or *U*. In this case, he/she would obtain a system of three nonlinear equations, which are significantly simpler than the case displayed before. Moreover, a manager may be satisfied by just rebalancing to his pre-specified optimal choice[2] (which as mentioned before may be an optimal allocation as derived under the frictionless markets hypothesis and any optimization objective); in that case, the system contains just two nonlinear equations. We display results based on this approach in the sixth section. The expression $\frac{C + A}{B}$ is derived as follows:

$$\begin{aligned} C &= K + \Pr\left\{T(U) > T(L) \middle| Y(0) = \pi\right\}k\left(U - \pi\right) + \Pr\left\{T(L) > T(U) \middle| Y(0) = \pi\right\}k\left(\pi - L\right) \\ &= K + v_1(\pi)k\left(U - \pi\right) + \left(1 - v_1(\pi)\right)k\left(\pi - L\right) \end{aligned}, \tag{3.24}$$

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[1] All calculations were performed via MATLAB's Symbolic Math Toolbox.

[2] This strategy has also been adopted in Korn [24] for a different problem to ours.

$$B = v_2(\pi) = 2\frac{\big(U\big(S(\pi)-S(L)\big)+\pi\big(S(L)-S(U)\big)+L\big(S(U)-S(\pi)\big)\big)}{\big(S(L)-S(U)\big)\sigma^2}$$
$$= \frac{2\pi}{\sigma^2} - \frac{2U}{\sigma^2}v_1(\pi) - \frac{2L}{\sigma^2}\big(1-v_1(\pi)\big) \tag{3.25}$$

and

$$A = v_3(\pi) = -v_1(\pi)h(U) - \big(1-v_1(\pi)\big)h(L) \tag{3.26}$$

with *h(y)* as in (3.23). Naturally, computations can be reduced even more if one just seeks a symmetric control band around his/her pre-specified rebalancing point (a single nonlinear equation).

The reader should also note that computations are significantly simplified by considering the transformed risky fraction process. For example for the scale function of the original process one would have

$$s(y) = \exp\left\{-\int^y \left[2\mu(\xi)/\sigma(\xi)^2\right]d\xi\right\} = \exp\left\{-\int^y \left[2\big(\mu - r - \sigma^2\xi\big)/\big(\sigma^2\xi(1-\xi)\big)\right]d\xi\right\} =$$
$$= 2\frac{\big(\log(y)-\log(y-1)\big)\big(\mu - r\big)}{\sigma^2} - \log(y-1) \tag{3.27}$$

and

$$S(y) = y^{-2p_1}(y-1)^{-2(p_1+1)} \tag{3.28}$$

with

$$p_1 = \frac{\mu - r}{\sigma^2}. \tag{3.29}$$

Due to the form of the scale function, calculation of the expected time to leave an interval or expected tracking error within the interval becomes significantly more tedious for the original process.

Minimization of discounted lifetime costs

In this section, we show how to solve the portfolio manager's tracking problem (2.11) by solving a quasi-variational inequality when the objective is to minimize expected discounted tracking error plus discounted transaction costs over lifetime. The problem could have been approached as in Suzuki and Pliska (2004) (they controlled the original risky fraction process) with a change in the objective function. Here, for computational simplicity we work with the transformed risky fraction process. It is also worth noting that impulse control problems similar to ours (deviating mainly in the objective function) have been applied in Cadenillas and Zapatero (1999) for optimal control of an exchange rate, Buckley and Korn (1998) and Baccarin (2002) for cash management and in Plehn-Dujowich (2005) for optimal price changes of a firm that faces menu costs.

Admissible rebalancing strategies

Since we want to minimize the functional *J* in (2.11), we should consider only those strategies for which *J* is well defined and finite. In order that

$$E\left[\int_0^\infty e^{-\beta t} g\big(y(t)\big)\right] = E\left[\int_0^\infty e^{-\beta t} e^{2(y(t)-\pi)}dt\right] - 2E\left[\int_0^\infty e^{-\beta t} e^{(y(t)-\pi)}dt\right] + \frac{1}{\beta} \tag{4.1}$$

be well defined and finite, we need that the two expected values on the right-hand-side be finite. It is straightforward to see that the condition

$$E\left[\int_0^\infty e^{-\beta t} e^{2(y(t)-\pi)} dt\right] < \infty \tag{4.2}$$

implies

$$E\left[\int_0^\infty e^{-\beta t} e^{(y(t)-\pi)} dt\right] < \infty \,. \tag{4.3}$$

Now in order that

$$E\left[\sum_{n=1}^\infty e^{-\beta\tau_n} c\big(y(\tau_n -), y_n\big) I_{\{\tau_n<\infty\}}\right] < \infty \tag{4.4}$$

we need that

$$E\left[\sum_{n=1}^\infty e^{-\beta\tau_n} I_{\{\tau_n<\infty\}}\right] < \infty \text{ and } E\left[\sum_{n=1}^\infty e^{-\beta\tau_n} \left|y(\tau_n -) - y(\tau_n)\right| I_{\{\tau_n<\infty\}}\right] < \infty \,. \tag{4.5}$$

To obtain the inequality on the left-hand-side, we need that

$$\forall T \in [0,\infty): \quad P\left\{\lim_{n\to\infty} \tau_n < T\right\} = 0. \tag{4.6}$$

To obtain the inequality on the right-hand-side, we need that

$$\lim_{T\to\infty} E\left[e^{-\beta T} y(T+)\right] = 0 \,. \tag{4.7}$$

Indeed, according to the formula of integration by parts (see section VI.38 of Rogers and Williams (1987)), for every $0 < s \le t < \infty$,

$$E\left[e^{-\beta t} y(t+)\right] - E\left[e^{-\beta s} y(s+)\right] = (\kappa-\beta) E\left[\int_s^t e^{-\beta u} y(u) du\right] + E\left[\sum_{n=1}^\infty e^{-\beta\tau_n} \left|y(\tau_{n-}) - y(\tau_n)\right| I_{\{\tau_n<\infty\}}\right] . \tag{4.8}$$

Thus, $E\left[\int_0^\infty e^{-\beta t} y(t) dt\right] < \infty$ and $E\left[\sum_{n=1}^\infty e^{-\beta\tau_n} \left|y(\tau_{n-}) - y(\tau_n)\right| I_{\{\tau_n<\infty\}}\right] < \infty$ imply[3] condition (4.7).

DEFINITION 4.1 (Admissible controls): We shall say that an impulse control is admissible if the conditions (4.2) (4.6) (4.7) are satisfied.

EXAMPLE 4.1 Let us consider, similar to Cadenillas and Zapatero (1999), the strategy of no intervention, that is $P\{\tau_1 = \infty\} = 1$. Then,

$$y(t) = y_0 \exp\left\{\left(\kappa - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \,. \tag{4.9}$$

Thus, according to Fubini's theorem,

$$E\left[\int_0^\infty e^{-\beta t} e^{2(y(t)-\pi)} dt\right] = e^{2(y_0-\pi)} \int_0^\infty \exp\left\{\left(e^{\kappa-\frac{1}{2}\sigma^2} - \beta\right) t\right\} dt \,. \tag{4.10}$$

Furthermore,

$$E\left[e^{-\beta T} y(T+)\right] = E\left[e^{-\beta T} y(T)\right] = y_0 e^{(\kappa-\beta)T} \,, \tag{4.11}$$

and obviously

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[3] The above conditions imply that, for every $0 \le t_n \uparrow \infty$, the sequence $E\left[e^{-\beta\tau_n} y(\tau_n +)\right]$ is a Cauchy sequence and therefore converges o a nonnegative number. The number must be zero otherwise $E\left[\int_0^\infty e^{-\beta t} y(t) dt\right] = \int_0^\infty E\left[e^{-\beta t} y(t)\right] dt = \infty$

$$E\left[\sum_{n=1}^{\infty} e^{-\beta\tau_n} \left|y(\tau_{n-}) - y(\tau_n)\right| I_{\{\tau_n<\infty\}}\right] = 0 \tag{4.12}$$

since there are no interventions. Hence the strategy of no intervention is admissible if and only if $\log\beta > \kappa - \frac{1}{2}\sigma^2$ and $\beta > \kappa$. Otherwise, the cost function would be infinity or not defined.

Solution via a quasi-variational inequality

Let *J*(.) denote the value function. That is for every $y_0 \in (-\infty,\infty)$,

$$J(y_0) := \inf_{\{(\tau_n, y_n)\}\in \mathrm{A}(y_0)} J\left(y_0, \{(\tau_n, y_n)\}\right) \tag{4.13}$$

and $\mathrm{A}(y_0)$ denotes the set of admissible strategies when the transformed risky fraction process starts from y_0. Define the minimum cost switching operator *M*, associated with any such function *J*(.) and the transaction cost function *c*(.,.) by taking

$$MJ(y) := \inf_{\tilde{y}} \left\{J(\tilde{y}) + c(y, \tilde{y})\right\}. \tag{4.14}$$

MJ(y) represents the value of the strategy that consists in choosing the best immediate intervention. Recall equation (2.9) satisfied by the transformed risky fraction process and define the second order partial differential operator *L* by taking

$$LJ(y) := \frac{1}{2}\sigma^2 J''(y) + \kappa J'(y) - \beta J(y). \tag{4.15}$$

Suppose there exists an optimal strategy for each initial point. Then, if the process starts at y_0 and follows the optimal strategy, the cost function associated with this optimal strategy is $J(y_0)$. On the other hand, if the process starts at y_0, selects the best immediate intervention, and then follows an optimal strategy, then the cost associated with this strategy is $MJ(y_0)$. Since the first strategy is optimal, its cost function is smaller than the cost function associated with the second strategy. Furthermore, these two costs are equal when it is optimal to jump. Hence, $J(y) \le MJ(y)$, with equality when it is optimal to intervene. In the continuation region, that is when the portfolio manager does not intervene, we must have $LJ(y) = -g(y)$,

with $g(y) := \lambda\left(\exp(y-\pi) - 1\right)^2$ denoting the tracking error rate.

By standard methods for impulse control problems (e.g. see Bensoussan (1982), Bensoussan and Lions (1984), Korn (1998, 1999)) we are led to the following quasi-variational inequality:

$$\min\left\{Lv(y) + g(y), Mv(y) - v(y)\right\} = 0. \tag{4.16}$$

Indeed, if *v* is a twice continuously differentiable function satisfying this qvi as well as the technical growth conditions depicted in the first part of this section, then

$$v(y) \le J\left(y, \left\{(\tau_n, y^n)\right\}\right) \tag{4.17}$$

for all $y \in \Re$ and all admissible strategies $\left\{(\tau_n, y^n)\right\}$. If, moreover, the strategy corresponding to *v* is admissible, then it is an optimal strategy and *v* (.) is identical to the value function *J*(.). The proof of this 'verification theorem' is lengthy, technical, and reasonably standard (e.g. see Korn (1998) or Bielecki and Pliska (2000)), so it will be omitted. The construction of the strategy corresponding to a solution *v* goes as follows. With $\tau_0 = 0$ and $Y(0-) = y_0$ one has

$$\tau_n := \inf\{t \geq \tau_{n-1} : v(y(t-)) = Mv(y(t-))\} \tag{4.18}$$

and

$$y^n = \arg\min_{\tilde{y} \in A}\{v(\tilde{y}) + c(y(\tau_n -), \tilde{y})\}. \tag{4.19}$$

Note that v defines a continuation region

$$C := \{y \in \Re : Mv(y) > v(y)\}, \tag{4.20}$$

as no transactions occur as long as $y(t) \in C$. But if $y(t) \in \partial C$ (e.g., if $y(t)$ hits the boundary of C), then a transaction immediately occurs, shifting the risky fraction process according to (4.19).

The infimum operator M, for our problem is

$$Mv(y) = \inf_{\tilde{y} \in \Re}\{v(\tilde{y}) + K + k|y - \tilde{y}|\} \tag{4.21}$$

thus qvi (4.16) becomes

$$0 = \min\left\{\frac{\sigma^2}{2}v''(y) + \kappa v'(y) - \beta v(y) + \lambda(\exp(y-\pi) - 1)^2, \inf_{y \in A}\{v(\tilde{y}) - v(y) + K + k|y - \tilde{y}|\}\right\} \tag{4.22}$$

We now explain how this qvi can be solved. The ordinary differential equation corresponding to (4.22) has a general solution of the form

$$v(y) = C_1 e^{-x_1 y} + C_2 e^{-x_2 y} + \tilde{h}(y) \tag{4.23}$$

where $\tilde{h}$ is the particular solution of the differential equation given by

$$\tilde{h}(y) = \lambda \frac{A_1 + A_2 \exp(y-\pi) + A_3 \exp(2(y-\pi))}{A_4} \tag{4.24}$$

where

$$\begin{aligned} A_1 &= 6\kappa(\sigma^2 - \beta) - 5\sigma^2\beta + 2(\sigma^2 + \beta^2) + 4\kappa^2 \\ A_2 &= 4\beta(2\sigma^2 + 2\kappa - \beta) \\ A_3 &= -\beta(2\kappa + \sigma^2 - 2\beta) \\ A_4 &= (2\kappa + 2\sigma^2 - \beta)\beta(\sigma^2 + 2\kappa - 2\beta) \end{aligned} \quad . \tag{4.25}$$

Here C_1 and C_2 are constants depending on boundary conditions and x_1, x_2 are formulated as follows

$$x_{1,2} = \frac{\kappa \pm \sqrt{\kappa^2 + 2\sigma^2\beta}}{\sigma^2}. \tag{4.25}$$

For most values of the data parameters, it can be shown that there exist four parameters satisfying $L<l<u<U$ such that the solution of the qvi (4.22) will be of the form

$$v(y) = \begin{cases} -ky + \{v(y) + kl + K\} & y \in (-\infty, L] \\ v(y) & y \in (L, U) \\ ky + \{v(y) - ku + K\} & y \in [U, \infty) \end{cases} \tag{4.26}$$

Here *(L, U)* is the continuation region. For $y \in (-\infty, L]$ one should immediately rebalance to $y=l$, and for $y \in [U, \infty)$ one should immediately rebalance to $y=u$. It remains to determine the values of the six parameters C_1, C_2, L, l, u and U. On that purpose, one should solve a system of six nonlinear equations. To derive these equations we note that the function $v(.)$ must be continuous at $y=l$, so

$$v(L) = -kL + v(l) + kl + K\ . \tag{4.27}$$

Similarly, we get a second equation for continuity at $y=u$,

$$v(U) = kU + v(u) - ku + K\ . \tag{4.28}$$

The derivatives at $y=L$ and U must be continuous, so

$$v'(L) = -k \tag{4.29}$$

and

$$v'(U) = k\ . \tag{4.30}$$

Since $\tilde{y} = l$ minimizes $v(\tilde{y}) + K + k(\tilde{y} - L)$ the first order necessary condition gives

$$v'(l) = -k \tag{4.31}$$

and similarly the final equation is

$$v'(u) = k\ . \tag{4.32}$$

The system of six equations can readily be solved by MATLAB for the six parameters; a detailed numerical illustration presented at the sixth section.

On the probability density function of the controlled risky fraction process

The distribution of wealth of an investor that follows a constant allocation strategy in frictionless Black-Scholes markets can be derived in a straightforward way. Indeed, wealth in this case is a geometric Brownian motion process and its unique transition probability density function satisfies the associated Kolmogorov backward differential equation. When frictions are accounted for, the risky fraction process is not constant; it evolves according to (2.6) in an interval and as soon as it reaches the boundaries, it is returned instantaneously to some optimally derived levels within the control band. To derive the distribution of wealth for an investor that performs a control band policy one needs to derive the distribution of the risky fraction process. Then, the probability density function of wealth is derived according to

$$f(w) = \int f(w|p) f(p) dp \tag{5.1}$$

where $f(w)$ is the probability density of wealth under the control band policy, $f(w|p)$ is the probability density of wealth for a given portfolio allocation p, and $f(p)$ stands for the probability density of the controlled risky fraction process. In this section, we derive the probability density function for the transformed risky fraction process (2.9); the corresponding density for the original process can then be derived by applying (2.8). A related result is given by Plehn-Dujowich (2005, theorem 6); this work though is based on a discrete-time approximation of the process within the band. Here we use results from Karlin & Taylor (1981, section 15.8) to derive the density of the transformed risky fraction process within a simple control band with a single rebalancing point and then proceed to the derivation of the density that corresponds to a control band with two rebalancing points.

Within a control band with a single rebalancing point π, we assume that process (2.9) starts from π and returns to it whenever the boundaries of the band (L, U) are reached. After such a return, the subsequent motion of the process behaves just like (2.9); this process is repeated at each attainment of level L or U. Thus, the controlled process consists of recurrent cycles of random time duration $D_1, D_2, D_3, \ldots,$ where the D_i are independently and identically distributed, with the same distribution as

$T_{L,U} = \min\{T_L, T_U\}$, the first exit time from the interval (L, U), starting from π. It follows that

$$E\left[D_i \middle| y(0) = \pi\right] = 2\left\{v_1(\pi)\int_\pi^U (S(U) - S(\xi))m(\xi)d\xi + (1 - v_1(\pi))\int_L^\pi (S(\xi) - S(L))m(\xi)d\xi\right\} \tag{5.2}$$

where $v_1(.), S(.), m(.)$ are given by (3.13), (3.10) and (3.12) respectively. Let $f(t, y)$ be the density function of $y(t)$. That is,

$$f(t, y)dy = \Pr\{y \le y(t) \le y + dy | y(0) = \pi\}. \tag{5.3}$$

The objective is to evaluate $a(y|L, \pi, U) = \lim_{t\to\infty} f(t, y)$, the limiting density of $y(t)$. To do this, we fix an interval $[y_1, y_2]$ with $L < y_1 < y_2 < U$ and define the process $\{I(t), t \ge 0\}$ by

$$I(t) = \begin{cases} 1 & if \quad y_1 \le y(t) \le y_2 \\ 0 & otherwise. \end{cases} \tag{5.4}$$

One may now split a typical cycle of length D_i into two parts: in one part $I(t)$ is on, in the other $I(t)$ is off. Now

$$\Pr\{I(t) = 1\} = E[I(t)] = \int_{y_1}^{y_2} f(t, y)dy \tag{5.5}$$

and from the renewal theorem (Karlin and Taylor, Section 7.C, chapter 5) it follows that

$$\lim_{t\to\infty} \Pr\{I(t) = 1\} = \frac{E\left[T_{y_1,y_2} \middle| y(0) = \pi\right]}{E\left[T_{L,U} \middle| y(0) = \pi\right]} . \tag{5.6}$$

The denominator of (5.6) is given by (5.2) whereas the nominator is derived by simply putting y_1, y_2 in place of L, U in (5.2). Since (5.6) holds for every y_1, y_2 in (L,U) we deduce that

$$a(y|L, \pi, U) = \lim_{t\to\infty} f(t, y) = \frac{2\{v_1(\pi)(S(U) - S(y)m(y)) + (1 - v_1(\pi))(S(y) - S(L)m(y))\}}{2\left\{v_1(\pi)\int_\pi^U S(U) - S(y)m(y)dy + (1 - v_1(\pi))\int_L^\pi S(y) - S(L)m(y)dy\right\}}. \tag{5.7}$$

Using (5.7), the limiting and stationary distribution for the controlled process within a band with two rebalancing points l,u is

$$a(y|L, l, u, L) = (v_1(u) + v_1(l))a(y|L, u, U) + (2 - v_1(u) - v_1(l))a(y|L, l, U). \tag{5.8}$$

Instead of working with the transformed risky fraction process one could try to derive the appropriate expressions for the controlled process in the original scale; in this case, computations of the denominators in (5.7) and (5.8) would have been significantly more difficult.

Numerical illustration

In this section, we provide numerical solutions for the control problems considered in the third and fourth parts of the article. In particular, we solve the simplified version of the "minimization of long run cost per unit time" problem treated in the third section. Here the portfolio manager just seeks to find the unknown outer boundaries of her/his control band; as soon as portfolio holdings reach the boundaries she/he rebalances to the pre-specified target levels. In this case one needs to solve a system of two nonlinear equations derived by forming $(C+A)/B$ from (3.24)-(3.26), taking the derivatives with respect to the unknown boundaries L and U, plugging in the constants that characterize market characteristics and investor's preferences and equating derivatives to zero. These two nonlinear equations appear to be quite complicated for standard nonlinear equation solvers that numerically calculate the system's gradient: in this case we had to write a computer program that implements the Newton-Raphson algorithm plus two functions that describe the nonlinear system and its gradient. For the "minimization of discounted lifetime costs" problem, we provide numerical solutions for the system of nonlinear equations (4.27)-(4.32) and derive the six unknowns: the two outer boundaries L and U the inner rebalancing points l and u and the two constants $C1$ and $C2$ in (4.23) that characterize the evolution of the value function within the control band. The equations involved in this case are much simpler compared to the previous ones and the system can be solved via MATLAB's *fsolve* routine that is a part of the optimization toolbox and numerically calculates the system's gradient. The six equations are also significantly simpler when compared with the ones presented in Suzuki and Pliska (2004) that treat a similar problem as the one presented in our fourth section. Here, Nagai's transformation on the dynamics of the risky fraction process derogates the hypergeometric functions involved in Suzuki and Pliska 's system of equations.

To compare with results derived for the simplified problem of the third section we also provide solutions for a simplified (non-optimal) version of the problem treated in the fourth section; here the portfolio manager just seeks for the two outer boundaries of her/his control band and rebalances to her/his pre-specified target levels while minimizing discounted lifetime tracking error plus transaction costs. Instead of (4.27)-(4.32), in this case one should solve the following system comprised by the value matching conditions at the target level and the smooth pasting conditions at the boundaries of the control band:

$$v(L) = -kL + v(\pi) + k\pi + K\,. \tag{6.1}$$

$$v(U) = kU + v(\pi) - k\pi + K\,. \tag{6.2}$$

$$v'(L) = -k \tag{6.3}$$

$$v'(U) = k\,. \tag{6.4}$$

The reader should note that the three nonlinear systems are quite complex and thus sensitive to the initial values provided as starting points for their solutions. For the sensitivity analysis conducted at the second part of this section, we first found appropriate initial values for a baseline experiment and then, for each perturbation of the parameters, we were plugging as initial values the outcomes of the previous run. MATLAB codes are available upon request from the authors.

A specific example

We first consider the following data for market characteristics and investor's preferences

κ=0.1, σ=0.2, λ=1, β=0.05, π=0.5, k=0.05, K=0.005.

For the simplified version of the "minimization of expected long run cost per unit time " problem we find

L_1= 0.435, U_1=0.5433

for the outer boundaries of the control band, after transforming back to the original scale. For the "minimization of discounted lifetime costs" problem of the fourth section we find

L=0.4338, l=0.4746 u=0.5023, U=0.5456, $C1$=-43.7633, $C2$= -0.0388

whereas for the simplified non-optimal version of the latter problem we have

L_2=0.4279, U_2=0.5463, $c1$=-43.7504, $c2$=-0.0371.

The errors are of the order 10^{-8}.

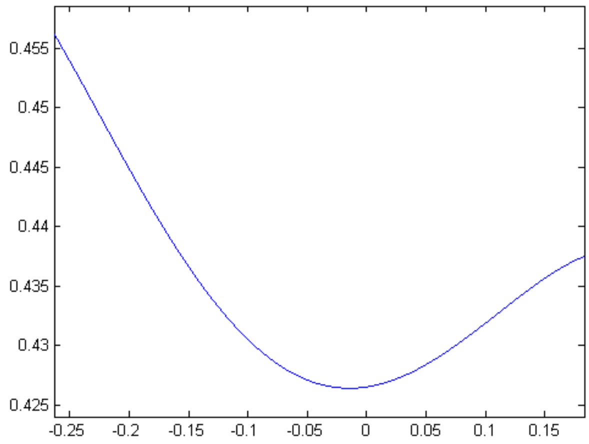

Figure 2. The value function for the minimization of discounted lifetime costs problem

Figure 2 depicts the value function corresponding to this parameter selection for the impulse control problem of the fourth section. The value function is depicted in the (transformed) continuation region (TL, TU). Outside this region, the value function is linear with slope $-k$ in the intervention region $(-\infty, TL]$ and a linear function with slope k in the intervention region $[TU, \infty)$. From (2.6) one observes that Merton's optimal proportion for the problem of maximizing the portfolio's exponential growth rate is an equilibrium point for the risky fraction process. In this example, Merton's proportion is much larger than the target proportion; thus, the risky fraction process is expected to force portfolio holdings to the right of the no-transaction region. For this reason the minimum point of the value function is located to the left of the target[4] proportion. When the investor intervenes on the weak side of the target (which corresponds to selling stock in this particular example) it is optimal to bring the portfolio levels much closer to the target than when she/he intervenes in the strong side. This difference between the target and Merton's proportion also causes asymmetry between the left and right part of the no-transaction region: the distance between the target asset proportion and the left boundary is almost double the one between the target and the right boundary. Starting from the target asset allocation, investor's holdings will reach the right boundary of the control band corresponding to minimizing discounted lifetime costs before the left boundary with probability 0.6581; for the simplified non-optimal version of this problem the corresponding probability is 0.6775. For the simplified problem of minimizing average long run cost per unit time the corresponding probability is 0.6636.

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[4] Zero in the tranformed scale corresponds to 0.5 in the original scale. This observation goes along the lines of Cadenillas and Zapatero [15] who treated a similar problem for the control of an exchange rate.

Sensitivity analysis

To conduct sensitivity analysis, we use as baseline values for the risky asset dynamics and investor's preferences the ones used in the example just before, and perturb each parameter separately to uncover how the optimal strategy is affected. Results corresponding to the full problem of the fourth section are depicted at figures 3 to 9 and tables 1,3,5,7,9,11. Results provided from the simplified versions of the problems considered in the third and fourth section, are depicted at tables 2,4,6,8,10,12. It should be underlined that the simplified problems produce very similar results to the "lifetime-problem" of the fourth section as far as the important "weak side" of the no transaction interval is concerned. Indeed, this is evident since the rebalancing points after reaching the upper boundary (resulting from the impulse control method of the fourth section) are very close to the target asset allocation levels and the no transaction intervals are very similar for the two methods.

Sensitivity of the control bands with respect to the cost parameters k and K, for the problem of the fourth section, is examined in figures 3 and 4 and tables 1 and 3. Solutions of $0<L<l<u<U<1$ are plotted as lines with each cost parameter varying in the horizontal axis. The intuition is clear: the investor rebalances more often with lower transaction costs. When fixed costs increase, it is optimal to wait longer before intervening although the sizes of interventions will be larger. When proportional costs increase, it is also optimal to wait longer but unlike when fixed costs increase, the interventions tend to be smaller. Tables 2 and 4 depict control bands for the simplified version of the problem presented in the third section (columns 2 and 3), for varying transaction cost levels (column 1), along with control bands for the simplified "minimization of discounted lifetime costs" problem, and constants $C1$ and $C2$ of equations (6.1)-(6.4) that characterize the value function within the control band. For all examined levels of K and k, the bands that correspond to the simplified problem of the third section are narrower than the ones derived via the system of (6.1)-(6.4); as K and k increase, the difference between bands seems to increase too.

In figure 5 and table 5 we show the effects of changes in volatility to the no-transaction bands derived from (4.27)-(4.32). As volatility increases, the no-transaction regions become wider and the magnitude of interventions becomes larger. Moreover, from table 6 we observe that no-transaction regions corresponding to the simplified problem of the third section tend to be more sensitive than the ones of the simplified "lifetime-problem" to changes in volatility. They are narrower for small values of volatility and wider for large values. In figure 6 and table 7, we analyze the effects of changes in the expected return of the risky asset on the optimal strategy of the investor that follows the impulse control method of the fourth section. As it increases, optimal interventions resulting from hitting the "weak-side" of the target tend to bring asset holdings at a level that is located lower than the target. The higher the pressure on the "weak side" of the target the sooner the investor should intervene; the opposite holds for the strong side of the target. For low levels of κ the controls bands of the simplified problem of the third section are wider than the ones of the simplified "lifetime problem"; this relationship is reversed as κ increases (table 8).

In accordance with intuition, it can be seen in figure 7 and table 9, that the bigger the values of the target asset mix the bigger the values of the four control parameters derived from 4.27-4.32. The target asset mix is not centered in the control bands; since the benchmark value we used for the expected return for the risky asset is quite high, the distances from p to u and from p to U are shorter than the ones from p to l and from p to L respectively. As p increases, the no transaction regions corresponding to the simplified problem of the third section get wider than the ones derived from the "lifetime-problem" (table 10). From figure 8 and table 11 we observe that as λ increases the investor becomes more concerned about tracking error relative to the target mix; thus, he width of the no-transaction region becomes narrower and small deviations form the target asset mix may induce rebalancing. Table 12 depicts that, as λ increases the no transaction region corresponding to the simplified problem of the third section gets narrower than the ones derived from the simplified "lifetime-problem". It should be remarked that λ is a parameter related to investor's wealth: the higher the amount of wealth invested the more sensitive the investor is to tracking error and the less important are the transaction costs for him. Finally, from figure 9 one observes that changes in the discount parameter β do not have a substantial effect on the optimal intervention strategy.

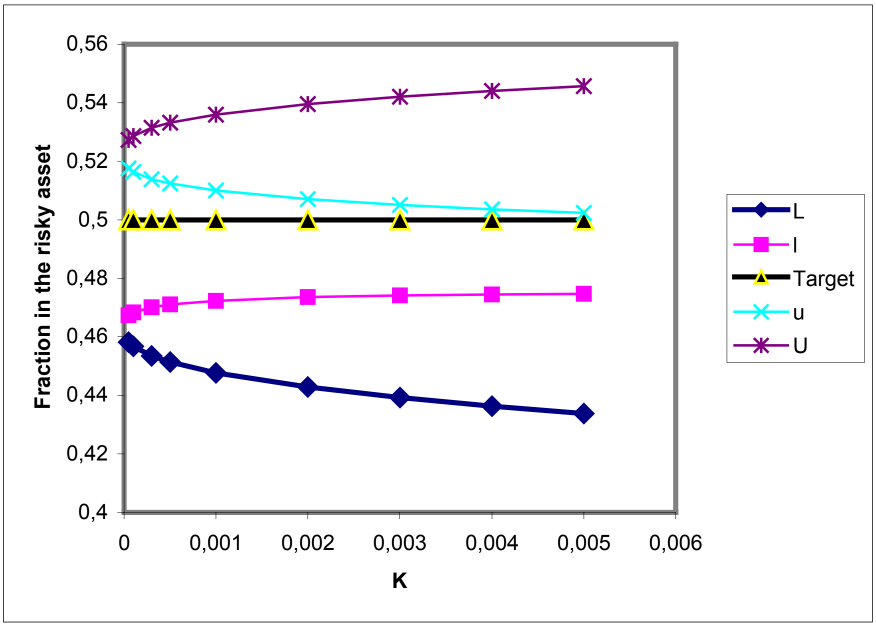

Figure 3. Sensitivity of optimal strategies to changes in *K*.

Table 1. Solutions to (4.27)-(4.32) for different levels of *K*.

K	L	l	u	U	C1	C2
0.0001	0.4567	0.4683	0.5163	0.5285	-43.8986	-0.0459
0.0005	0.4534	0.4701	0.5138	0.5314	-43.8738	-0.0447
0.001	0.4513	0.4710	0.5123	0.5331	-43.8555	-0.0436
0.002	0.4476	0.4723	0.5099	0.5359	-43.8269	-0.0421
0.003	0.4428	0.4735	0.5070	0.5395	-43.8032	-0.0408
0.004	0.4392	0.4741	0.5051	0.5420	-43.7824	-0.0398
0.005	0.4363	0.4744	0.5035	0.5440	-43.7633	-0.0388

Note. κ=0.1, σ=0.2, λ=1, β=0.05, π=0.5, k=0.05.

Table 2. Control bands derived from the simplified problems of section 3 (L1, U1) and section 4 (L2, U2) for different levels of *K*. C1 and C2 correspond to the constants in equations (6.1)-(6.4).

K	L1	U1	C1	C2	L2	U2
0.0001	0.4438	0.5343	-43.8499	-0.0421	0.4407	0.5355
0.0005	0.4428	0.5356	-43.8399	-0.0416	0.4393	0.5370
0.001	0.4417	0.5369	-43.8281	-0.0410	0.4377	0.5386
0.002	0.4397	0.5390	-43.8064	-0.0398	0.4348	0.5411
0.003	0.4380	0.5407	-43.7865	-0.0388	0.4323	0.5431
0.004	0.4364	0.5421	-43.7679	-0.0379	0.4299	0.5448
0.005	0.4350	0.5433	-43.7504	-0.0371	0.4279	0.5463

Note. κ=0.1, σ=0.2, λ=1, β=0.05, π=0.5, k=0.05.

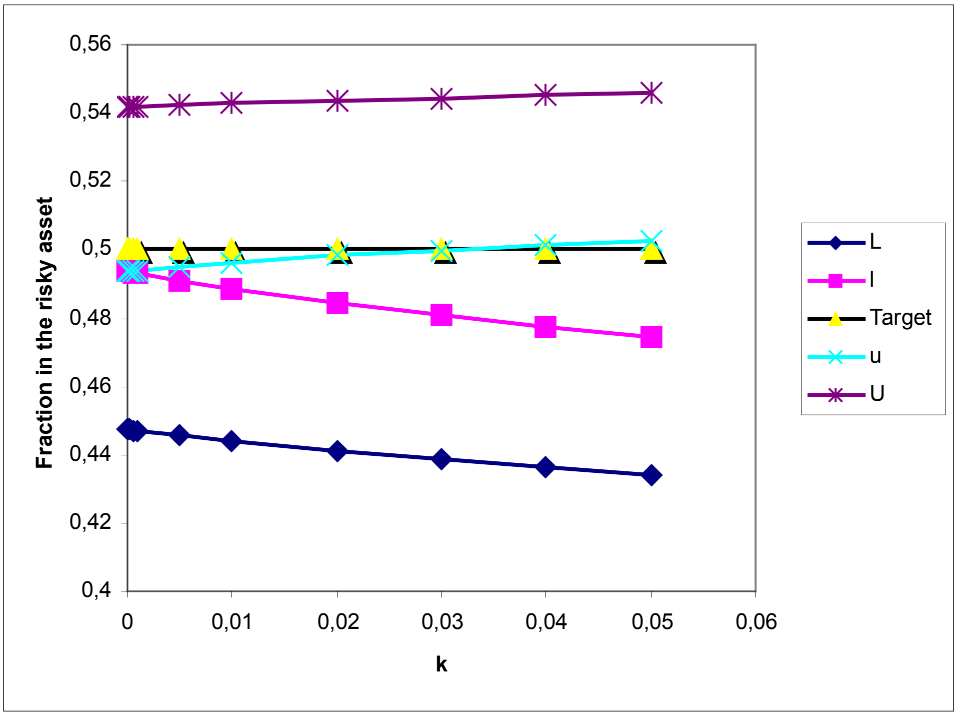

Figure 4. Sensitivity of optimal strategies to changes in *k*.

Table 3. Solutions to (4.27)-(4.32) for different levels of *k*.

k	**L**	**l**	**u**	**U**	**C1**	**C2**
0.00005	0.4472	0.4936	0.4936	0.5417	-43.9409	-0.0485
0.0001	0.4472	0.4935	0.4936	0.5417	-43.9407	-0.0485
0.0005	0.4471	0.4933	0.4937	0.5417	-43.9391	-0.0484
0.001	0.4469	0.4930	0.4939	0.5418	-43.9369	-0.0483
0.005	0.4455	0.4909	0.4950	0.5422	-43.9205	-0.0473
0.01	0.4440	0.4885	0.4963	0.5427	-43.9008	-0.0461
0.02	0.4411	0.4843	0.4983	0.5436	-43.8636	-0.0443
0.03	0.4385	0.4808	0.4998	0.5444	-43.8287	-0.4210
0.04	0.4360	0.4776	0.5011	0.5451	-43.7954	-0.0404
0.05	0.4338	0.4746	0.5023	0.5456	-43.7633	-0.0388

Note. κ=0.1, σ=0.2, λ=1, β=0.05, π=0.5, K=0.005.

Table 4. Control bands derived from the simplified problems of section 3 (L1, U1) and section 4 (L2, U2) for different levels of *k*. C1 and C2 correspond to the constants in equations (6.1)-(6.4).

k	L1	U1	C1	C2	L2	U2
0.00005	0.4475	0.5414	-43.9368	-0.0482	0.4467	0.5421
0.0001	0.4475	0.5414	-43.9366	-0.0482	0.4467	0.5421
0.0005	0.4474	0.5414	-43.9350	-0.0481	0.4466	0.5421
0.001	0.4473	0.5415	-43.9329	-0.0480	0.4464	0.5421
0.005	0.4461	0.5417	-43.9166	-0.0469	0.4448	0.5426
0.01	0.4447	0.5419	-43.8967	-0.0457	0.4428	0.5430
0.02	0.4420	0.5423	-43.8581	-0.0433	0.4389	0.5440
0.03	0.4395	0.5426	-43.8209	-0.0411	0.4351	0.5448
0.04	0.4372	0.5430	-43.7851	-0.0390	0.4315	0.5456
0.05	0.4350	0.5433	-43.7504	-0.0371	0.4279	0.5463

Note. κ=0.1, σ=0.2, λ=1, β=0.05, π=0.5, K=0.005.

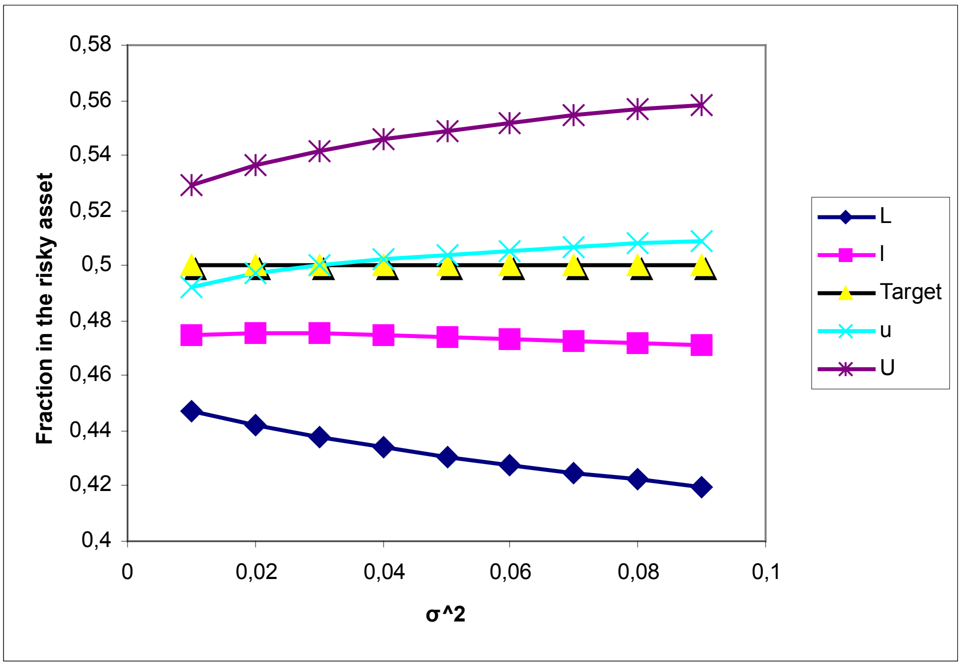

Figure 5. Sensitivity of optimal strategies to changes in σ.

Table 5. Solutions to (4.27)-(4.32) for different levels of σ^2.

σ^2	L	l	u	U	C1	C2
0.01	0.4470	0.4746	0.4919	0.5290	-50.2393	-0.0002
0.04	0.4419	0.4755	0.4970	0.5364	-47.7612	-0.0045
0.07	0.4247	0.4723	0.5067	0.5542	-39.3635	-0.1472
0.09	0.4198	0.4708	0.5088	0.5584	-37.1212	-0.2434

Note. κ=0.1, λ=1, β=0.05, π=0.5, k=0.05, K=0.005.

Table 6. Control bands derived from the simplified problems of section 3 (L1, U1) and section 4 (L2, U2) for different levels of σ^2. C1 and C2 correspond to the constants in equations (6.1)-(6.4).

σ^2	L1	U1	C1	C2	L2	U2
0.01	0.4734	0.5188	50.2288	-0.0001	0.4384	0.5299
0.04	0.4350	0.5433	-43.7504	-0.0371	0.4279	0.5463
0.07	0.4213	0.5530	-39.3404	-0.1432	0.4184	0.5554
0.09	0.4074	0.5622	-37.0920	-0.2377	0.4130	0.5599

Note. κ=0.1, λ=1, β=0.05, π=0.5, k=0.05, K=0.005.

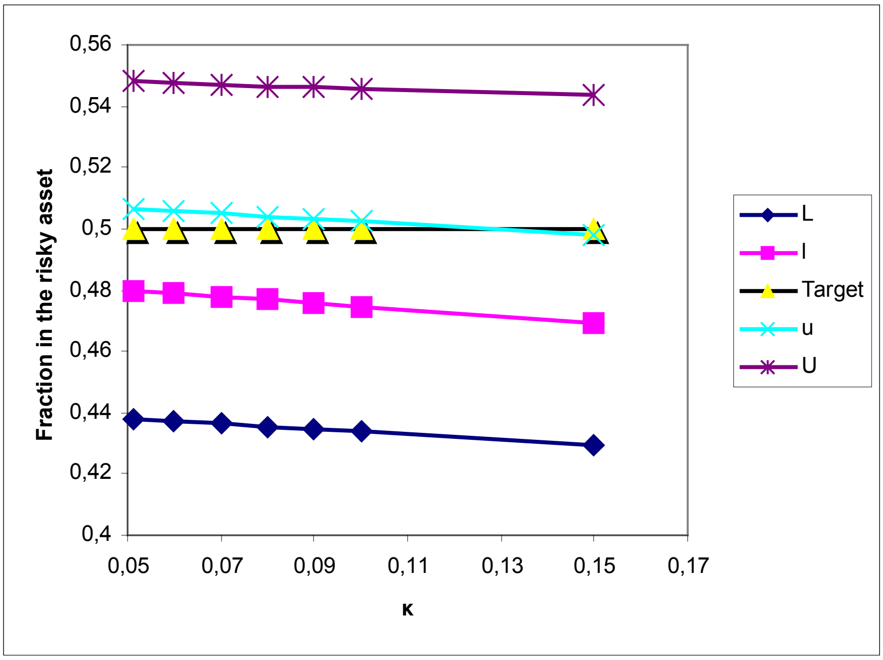

Figure 6. Sensitivity of optimal strategies to changes in κ.

Table 7. Solutions to (4.27)-(4.32) for different levels of *κ*.

κ	L	l	u	U	C1	C2
0.05	0.4377	0.4796	0.5065	0.5479	-104.8479	-0.2518
0.06	0.4371	0.4788	0.5057	0.5475	-79.4152	-0.1779
0.07	0.4362	0.4777	0.5048	0.5470	-63.5884	-0.1195
0.08	0.4354	0.4767	0.5040	0.5465	-54.2425	-0.0812
0.09	0.4346	0.4757	0.5031	0.5461	-48.0985	-0.0558
0.10	0.4338	0.4746	0.5023	0.5456	-43.7633	-0.0388
0.15	0.4294	0.4694	0.4981	0.5436	-33.1793	-0.0073

Note. σ=0.2, λ=1, β=0.05, π=0.5, k=0.05, K=0.005.

Table 8. Control bands derived from the simplified problems of section 3 (L1, U1) and section 4 (L2, U2) for different levels of *κ*. C1 and C2 correspond to the constants in equations (6.1)-(6.4).

κ	L1	U1	C1	C2	L2	U2
0.05	0.4268	0.5524	-104.834	-0.2477	0.4337	0.5489
0.06	0.4293	0.5500	-79.4014	-0.1745	0.4327	0.5484
0.07	0.4311	0.5479	-63.5743	-0.1166	0.4316	0.5479
0.08	0.4327	0.5461	-54.2292	-0.0787	0.4304	0.5473
0.09	0.4339	0.5446	-48.0853	-0.0538	0.4292	0.5468
0.10	0.4350	0.5433	-43.7504	-0.0371	0.4279	0.5463
0.15	0.4385	0.5383	-33.1691	-0.0065	0.4211	0.5442

Note. σ=0.2, λ=1, β=0.05, π=0.5, k=0.05, K=0.005.

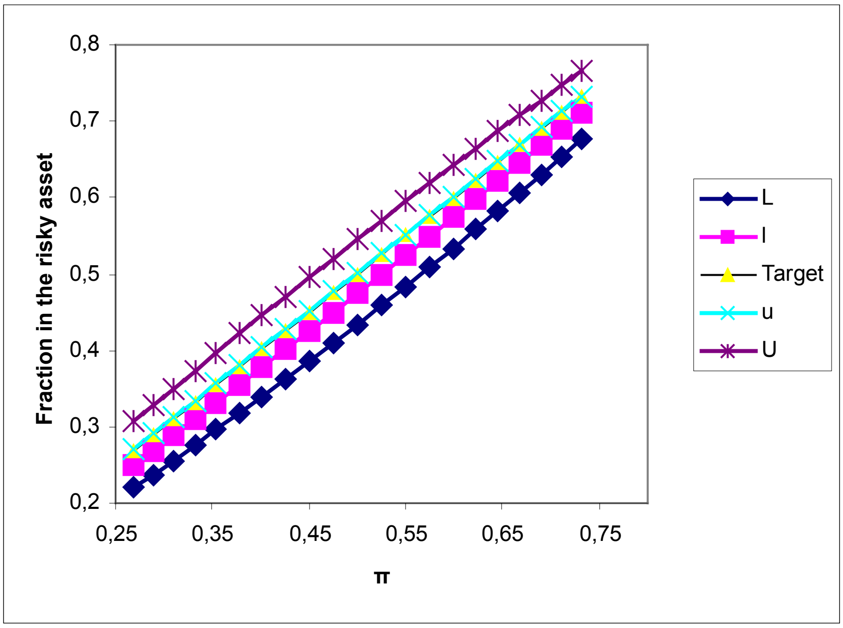

Figure 7. Sensitivity of optimal strategies to changes in *p*.

Table 9. Solutions to (4.27)-(4.32) for different levels of *p*.

p	**L**	**l**	**u**	**U**	**C1**	**C2**
0.2689	0.2198	0.2494	0.2707	0.3064	-69.1888	-0.0002
0.2890	0.2375	0.2686	0.2909	0.3281	-66.0912	-0.0003
0.3100	0.2561	0.2887	0.3120	0.3505	-63.1321	-0.0005
0.3318	0.2756	0.3097	0.3338	0.3736	-60.3056	-0.0009
0.3543	0.2960	0.3315	0.3564	0.3973	-57.6056	-0.0015
0.3775	0.3172	0.3540	0.3797	0.4214	-55.0266	-0.0025
0.4013	0.3393	0.3772	0.4035	0.4460	-52.5631	-0.0044
0.4255	0.3620	0.4010	0.4278	0.4708	-50.2097	-0.0076
0.45016	0.3854	0.4252	0.4524	0.4958	-47.9618	-0.0130
0.4750	0.4094	0.4498	0.4773	0.5208	-45.8145	-0.0225
0.5000	0.4338	0.4746	0.5023	0.5456	-43.7633	-0.0388
0.5249	0.4585	0.4996	0.5272	0.5703	-41.8042	-0.0670
0.5498	0.4834	0.5246	0.5521	0.5946	-39.9324	-0.1157
0.5744	0.5084	0.5495	0.5767	0.6185	-39.0283	-0.1521

Note. κ=0.1, σ=0.2, λ=1, β=0.05, k=0.05, K=0.005.

Table 10. Control bands derived from the simplified problems of section 3 (L1, U1) and section 4 (L2, U2) for different levels of p. C1 and C2 correspond to the constants in equations (6.1)-(6.4).

p	L1	U1	C1	C2	L2	U2
0.2689	0.2393	0.2936	-69.1682	-0.0002	0.2158	0.3070
0.2890	0.2571	0.3153	-66.07	-0.0003	0.2332	0.3286
0.3100	0.2757	0.3379	-63.1134	-0.0005	0.2515	0.3511
0.3318	0.2949	0.3614	-60.2878	-0.0008	0.2708	0.3742
0.3543	0.3147	0.3856	-57.5886	-0.0014	0.2910	0.3978
0.3775	0.3349	0.4106	-55.0103	-0.0024	0.3121	0.4221
0.4013	0.3555	0.4361	-52.5475	-0.0042	0.3339	0.4466
0.4255	0.3761	0.4623	-50.1949	-0.0072	0.3565	0.4715
0.45016	0.3965	0.4889	-47.9476	-0.0125	0.3798	0.4964
0.4750	0.4163	0.5159	-45.801	-0.0215	0.4036	0.5214
0.5000	0.4350	0.5426	-43.7504	-0.0371	0.4279	0.5463
0.5249	0.4519	0.5710	-41.7917	-0.064	0.4525	0.5709
0.5498	0.4669	0.5987	-39.9206	-0.1105	0.4774	0.5952
0.5695	0.4700	0.6315	-38.4843	-0.171	0.4974	0.6143

Note. κ=0.1, σ=0.2, λ=1, β=0.05, k=0.05, K=0.005.

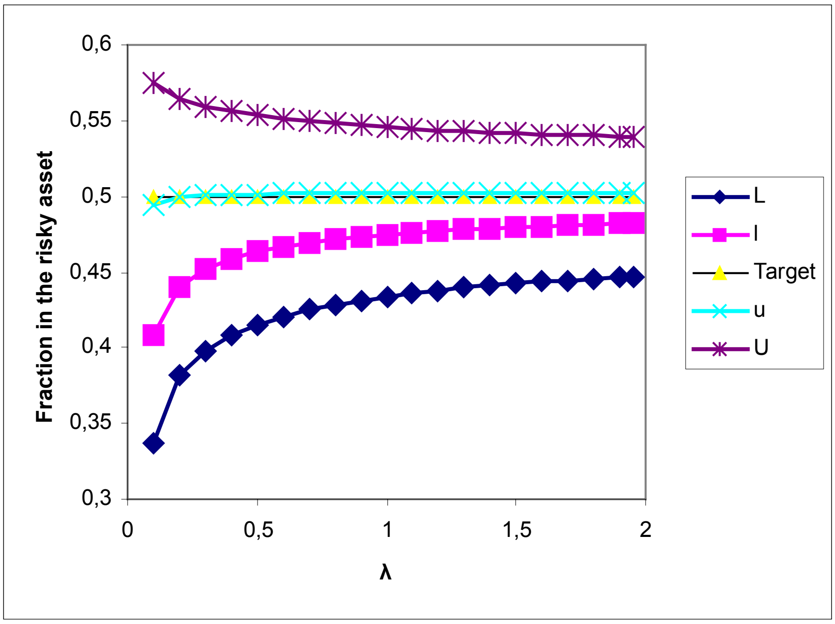

Figure 8. Sensitivity of optimal strategies to changes in λ.

Table 11. Solutions to (4.27)-(4.32) for different levels of λ.

λ	L	l	u	U	C1	C2
0.5	0.4153	0.4634	0.4985	0.5534	-21.7713	-0.0148
1	0.4338	0.4747	0.5023	0.5457	-43.7633	-0.0388
1.5	0.4423	0.4795	0.5025	0.5416	-65.7774	-0.0651
2	0.4476	0.4823	0.5025	0.5389	-87.8025	-0.0923

Note. κ=0.1, σ=0.2, β=0.05, π=0.5, k=0.05, K=0.005.

Table 12. Control bands derived from the simplified problems of section 3 (L1, U1) and section 4 (L2, U2) for different levels of λ. C1 and C2 correspond to the constants in equations (6.1)-(6.4).

λ	L1	U1	C1	C2	L2	U2
0.5	0.4079	0.5587	-21.7621	-0.0134	0.4051	0.5541
1	0.4315	0.5459	-43.7504	-0.0371	0.4279	0.5463
1.5	0.4432	0.5419	-65.7622	-0.0631	0.4380	0.5422
2	0.4468	0.5391	-87.7859	-0.0903	0.4441	0.5394

Note. κ=0.1, σ=0.2, β=0.05, π=0.5, k=0.05, K=0.005.

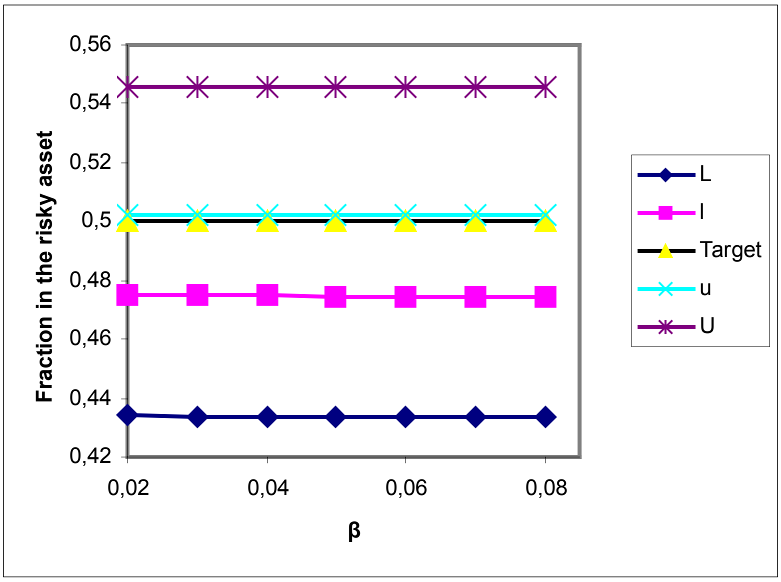

Figure 9. Sensitivity of optimal strategies to changes in β.

Concluding remarks

In this paper, we derive optimum portfolio strategies when transaction costs are taken into account. Instead of direct optimization with respect to investor's objectives, we propose that the investor may track a constant allocation policy as derived under the frictionless market hypothesis by applying a loss function for the tracking error that reflects her/his preferences. To illustrate our methods, we use quadratic loss and derive control bands for investors that either aim to minimize long run tracking error plus transaction cost per unit time or aim to minimize discounted tracking error plus transaction costs over an infinite horizon. The adopted methods are different for the two objectives but Nagai's transformation appears to be a valuable tool for both cases. In the first case, it reduces drastically the expressions for the expected duration of a transaction cycle and the expected tracking error during a cycle. In the second case, it simplifies significantly the system of nonlinear equations that one needs to solve for the derivation of the optimal control parameters.

To our knowledge, this is the first research effort that compares "classic" control band policies derived by minimizing expected discounted transaction costs plus tracking error over lifetime, to the ones derived by minimizing expected long tracking error plus transaction cost per unit time. The latter modeling strategy can be straightforwardly adapted to problems where only the former has been applied. To name a few, it can be applied to the problem of tracking a target level for an exchange rate as in Cadenillas and Zapatero (1999), to the cash management problem, or to the problem of optimal price changing for a firm that faces menu costs, as in Plehn-Dujowich (2005). Moreover, control band policies derived by discounted lifetime minimization require market coefficients and transaction costs coefficients to be constant over lifetime for the bands to be valid. With the objective of the third section, one may derive different valid bands for different transaction cost parameters; these bands are valid as long as market coefficients are stable for a long period.

After deriving control bands for tracking constant portfolio allocations in the presence of constant and proportional transaction costs, a research question that emerges next, is related to constructing control bands for tracking moving targets. For example, a portfolio manager may wish to gradually reduce portfolio holdings in risky assets as the investor's life evolves. Standard methods that minimize expected discounted transaction costs plus tracking error over lifetime cannot be straightforwardly adapted to solve this problem. On the contrary, we believe that methods similar to the ones presented in the third section can be relatively easily adapted to cover this case. We plan to develop this idea in a forthcoming publication.

Our sensitivity analysis of the sixth section confirms and extends economic intuition. However there remains (at least) an issue where our study did not develop a satisfactory economic understanding: it is the explanation of the differences between the control bands derived by the two alternative minimization criteria. We hope that this issue will be resolved in future research. If not then our sensitivity study illustrates a good reason to develop such models: as Suzuki and Pliska (2004) point out, intuition has limits, in which case model outputs often provide the best guidance for making economic and financial decisions.

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