Uncertainty Aversion, Robust Control and Asset Holdings

Giannis Vardas and Anastasios Xepapadeas[1]

University of Crete, Department of Economics

Rethymno 74 100, Greece

E-mail: vardasg@econ.soc.uoc.gr, xepapad@econ.soc.uoc.gr

Version: February 2004

Optimal portfolio rules are derived under uncertainty aversion by formulating the portfolio choice problem as a robust control problem. The robust portfolio rule indicates that the total holdings of risky assets as a proportion of the investor's wealth could increase as compared to the holdings under the Merton rule, which is the standard risk aversion case. In particular, with two risky assets and one risk-free asset, we show that uncertainty aversion could lead to an increase in the holdings of the one risky asset, accompanied by a reduction in the holdings of the other risky asset. Furthermore, in the optimal robust portfolio the investor may increase the holdings of the asset for which there is or less ambiguity, and reduce the holdings of the asset for which there is more ambiguity, a result that might provide an explanation of the home bias puzzle.

[1] We would like to thank William Brock and Angelos Kanas for helpful comments and suggestions.

Key Words: Uncertainty Aversion, Model Misspecification, Robust Control, Portfolio Choice Models.

Subject Classification: JEL Classification: G11, D81

1. INTRODUCTION

The assumption that asset prices are generated by a geometric Brownian motion and that consumers-investors know the associated true probability law has been central in the analysis of portfolio choice models (Merton [15], [16]). In economics the assumption that the true probability distribution associated with an event is known, and thus the expected utility framework can be used as a methodological framework, has come under some criticism because of its failure to explain certain "puzzles" such as the observed equity premium puzzle, or the investors home-bias puzzle. In trying to explain these puzzles attention has been focused on the case where the decision maker faces pure uncertainty in the Knightian sense, or ambiguity, and its preference relationship is characterized by uncertainty aversion (Gilboa and Schmeidler [12])

There are two main approaches to the problem of choices when the agent is assumed to be uncertainty averse. In the first, the multiple priors model, the decision maker may formulate his/her objective by attaching a probability, say e, to a baseline prior and a probability $(1 - e)$ to the infimum of a family of the disturbed priors around the baseline one. This is the so-called e-contamination approach (Epstein and Wang [9]), which is consistent with uncertainty or ambiguity aversion.[2] In the second approach the agent considers model misspecification. In this approach the decision maker is unsure about his/her model, in the sense that there is a group of approximate models that are also considered as possibly true given a set of finite data. These approximate models are obtained by disturbing a benchmark model, and the admissible disturbances reflect the set of possible probability measures that the decision maker is willing to consider. The resulting problem is one of robust dynamic control, where the objective is to choose a rule that will work under a range of different model specifications. This methodology provides another tractable way to incorporate uncertainty aversion (e.g. Hansen and Sargent, [19], [20], [21], [23], Hansen et al [24]).

Portfolio choice theory has been a prominent area of application of the above approaches[3] (e.g. Dow and Werlang [5], Epstein and Wang [9], Chen and Epstein [7], Epstein and Miao [8], Uppal and Wang [31],

[2] Chen and Epstein [7], introduce ambiguity aversion to recursive multiple-prior models of utility by considering $\kappa - Ignorance$ which is a concept that allows differentiation between ambiguous and pure risk cases.

[3] Monetary policy can be regarded as the initial area of application of these approaches (e.g., Brainard, [1] Hansen and Sargent [23], Onatski and Stock [17], Onatski and Williams [18], Soderstrom [29]). See also Brock and Durlauf [2], Brock, Durlauf and West [3] for similar approaches to policy design and policy evaluation under uncertainty. Another area of interest is environmental and resource management where uncertainty aversion can be used to formulate the concept of the Precautionary Principle (Brock and Xepapadeas [4], Roseta-Palma and Xepapadeas [27])

Maenhout [25], Pathak [26], Liu [13], [14]). The idea behind the use of robust control methods in optimal portfolio choice is that the investor faces model uncertainty regarding the assets' price processes. Thus, although the available data used to estimate the probability law characterizing the evolution of asset prices allow for the estimation of a benchmark model, there is a set of alternative models describing the evolution of asset prices, which is also consistent with the data and could be regarded as possibly true. In this set up the investor seeks a portfolio rule that will work, in the sense of maximizing utility, under a range of different model specifications of the assets' price equations.

In recent attempts to study Merton's basic optimal portfolio choice problem in the context of the Hansen-Sargent robust control methodology, Maenhout [25] considers a two asset problem, a risky asset and a riskless asset for an investor maximizing a CRRA utility function. The derived robust portfolio rule is an adjusted Merton rule. When there is no preference for robustness, or to put it differently, there is no concern for model misspecification, which implies that the so-called robustness parameter $\theta \to \infty$, this rule tends to Merton's rule.[4] Uppal and Wang [31] extend the problem to

[4] The robustness parameter θ can be interpreted as the Lagrangian multiplier associated with an entropy constraint, which determines the maximum specification error in the asset price equation that the investor is willing to accept (Hansen and Sargent [21]). As such it is a fixed parameter and characterizes preferences consistent with Gilboa and Schmeidler's axiomatization of uncertainty aversion. When $\theta \to \infty$ there is no concern about model misspecification and we are in the usual risk aversion framework.

the n asset case and derive a generalized robust portfolio rule, which allows for different degrees of robustness associated with different subsets of the asset price process. A central result underlying the recent robust control literature in the portfolio selection context (Maenhout [25], Uppal and Wang [31]) suggests that model uncertainty implies cautiousness in the sense that the investor, under uncertainty aversion, will invest a smaller share of his/her wealth in the risky assets relative to the share implied by the standard Merton rule under risk aversion. In more general terms, model uncertainty seems to have been associated in the earlier literature with some kind of cautious or conservative behavior,[5] although more recent results in the area of monetary policy analysis under uncertainty seem to provide mixed findings, that is aggressiveness or robustness depending on the structure of the model.[6]

The present paper attempts to derive optimal portfolio rules under uncertainty aversion by following Hansen and Sargent's approach and formulating the portfolio choice problem as a robust control problem. We derive portfolio rules for the cases of two and multiple, $i = 1, ..., n$, risky assets. We also allow for uncertainty aversion, or preference for robustness, both with respect to the joint distribution of the assets, and, in the general model, with respect to the distribution of each risky asset. Our portfolio

[5] For example Brainard's [1] results suggest caution in the face of model uncertainty in a Bayesian framework.

[6] See for example Onatski and Williams [18] and the papers cited by them.

rules are parametrized by the robustness parameter θ.[7] We show that as $\theta \to \infty$ the robust portfolio rule tends to Merton's rule in accordance with Maenhout's results. However, under uncertainty aversion, the associated robust portfolio rule indicates that the total holdings of risky assets as a proportion of the investor's wealth is not always smaller as compared to the holdings under the Merton rule, which is the risk aversion case, and which is equivalent to no concerns about model misspecification and $\theta \to \infty$. This result seems to depart from the belief that uncertainty, or ambiguity aversion, and the associated robust control methods might result in more cautiousness or conservatism regarding portfolio choices, in the sense that holdings of the "risky - ambiguous" assets are reduced relative to the pure risk case. We derive conditions under which such an increase in the total holdings of risky assets takes place, which are independent of the form of the utility function and the value of the robustness parameter θ.[8] With

[7] When we allow for different levels of ambiguity associated with different assets, the portfolio rule is parametrized by the vector $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$.

[8] The independence from the value of θ is desirable since θ is basically exogenous. There have been attempts to eliminate θ from the portfolio rule as shown by Pathak [26]. Maenhout's transformation of the robustness parameter θ to a time dependent function, with θ being proportional to the value function of the robust problem portfolio, in order to make the portfolio rule independent of θ, breaks down the consistency of preferences with Gilboa and Schmeidler's axiomatization of uncertainty aversion, and transforms the robust rule to Merton's rule with a lower drift of the asset price equations. Pathak shows also that the Uppal and Wang [31] rule depends on a normalization factor which is taken to be proportional to the value function as in Maenhout's transformation.

two risky assets we show that under certain condition uncertainty aversion could induce an increase in the holdings of the one risky asset as compared to risk aversion. When this happens the holding of the other asset is always reduced.[9]

Next we examine the case when the investor has different preferences for robustness for each asset, or different levels of ambiguity regarding each asset. For the multiple asset case we derive again conditions under which the total holdings of assets increase under uncertainty aversion. For the two asset case we show that in the optimal robust portfolio the investor may increase the holdings of the asset for which there is less concern about model misspecification (high θ), or less ambiguity, and reduce the holding of the asset for which there is more concern about model misspecification (low θ), or more ambiguity, relative to the risk aversion case ($\theta = \infty$). If

Pathak [26] avoids the transformation that endogenizes the robustness parameter in terms of the value function by directly determining the worst possible distortion in terms of an instantaneous relative entropy constraint. In this case the robust portfolio problem is simply Merton's problem with a reduced mean return, the reduction defined in terms of the worse possible distortion. (See also Chen and Epstein [7]). It seems that since the exogeneity of θ is required in order for the problem to be consistent with uncertainty aversion, robust portfolios are parametrized by θ. To estimate θ in order to fully characterize the robust portfolio, Hansen and Sargent [19] suggest the use of detection error probabilities.

[9] The extent to which the net result of these two opposite effects will increase or decrease total holdings of risky assets depends on the conditions of the previous result which characterize the behavior of total holdings.

we associate the asset with less ambiguity with home assets and the asset with more ambiguity with foreign assets, this result could be regarded as an additional explanation of the home bias puzzle.[10]

2. ROBUST PORTFOLIO CHOICES

We consider a market which consists of one riskless asset whose price evolves accordingly to:

$$dS(t) = rS(t)dt \qquad S(0) = S_0, \quad t \geq 0,$$

where r denotes the risk-free rate of return, and $i = 1, ..., n$ risky assets. Denoting by $(\alpha_1, \alpha_2, ..., \alpha_n)$ the drift rates, or mean rates of return, and by $(\sigma_1, \sigma_2, ..., \sigma_n)$ the volatility rates, the evolution of the prices $P = [diag(P_1, P_2, ..., P_n)]$ of the n assets, is given by:

$$dP = P\mathcal{A}dt + P\Sigma R dB \tag{1}$$

[10] There have been a number of arguments attempting to explain the home bias puzzle. Strong and Xu [30] explain the puzzle on the basis of optimism of fund managers towards their home equity market. Serrat [28] considers nontraded goods to operate as factors that shift the marginal utility of traded goods. This entails dynamic hedging policies which in turn are consistent with the home bias puzzle, while French and Poterba [11] consider information costs as an explanation of the puzzle. Our approach of providing a partial explanation to the puzzle through uncertainty aversion is along the lines of the approach used by Uppal and Wang[31] and Pathak [26].

where $\mathcal{A}, \Sigma$, are $n \times n$ diagonal matrices with diagonal elements α_i, σ_i respectively, R [11] is a matrix such that $\Sigma R (\Sigma R)^T$ is equal to the variance - covariance matrix and $B = [B_1, B_2, ..., B_n]^T$ [12] are n independent Brownian processes, defined on an underlying probability space $(\Omega, \mathcal{F})$, with measure $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2 \otimes ... \otimes \mathcal{P}_n$.

Merton's solution ([15], [16]) of the optimal portfolio allocation problem for an infinite time horizon and n risky assets, determines the optimal portfolio weights, w_i, that is the fraction of the investor's total wealth W allocated to asset i as:

$$\begin{aligned} w_i W &= A \sum_{j=1}^{n} \Upsilon_{ij}^{-1} (\alpha_i - r) \qquad i = 1, ..., n \qquad (2) \\ A &= -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C) \frac{\partial C}{\partial W}} \end{aligned}$$

where V is the value function of the problem, V_W and V_{WW} are the first and the second partial derivatives respectively with respect to the wealth W, Υ^{-1} is the inverse of the variance - covariance matrix Υ, and $U(C)$ is a standard utility function.

Following Hansen and Sargent [22] and Hansen et al. [24], model (1) is regarded as a benchmark model. If the consumer-investor was sure about the benchmark model, then there would be no concerns about robustness regarding model misspecification. Otherwise, these concerns can be reflected by a family of stochastic perturbations. Because there are n in-

[11] A typical element of matrix R can be seen in the next section when we examine the case of two risky assets.

[12] Superscript T denotes the transpose of a matrix.

dependent Brownian motions,[13] each one can be perturbed separately, so that:

$$B_i(t) = \hat{B}_i(t) + \int_0^t h_i(s)ds \quad , \ i = 1, ..., n \tag{3}$$

where $\{\hat{B}_i(t) : t \geq 0\}$ are Brownian motions and $\{h_i(t) : t \geq 0\}$ are measurable drift distortions. Therefore the probabilities implied by (1) are distorted. The measure $\mathcal{P}$ is replaced by another probability measure $\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2, ..., \otimes \mathcal{Q}_n$. As shown by Hansen et al. [24], the discrepancy between the distribution $\mathcal{P}$ and $\mathcal{Q}$ is measured as the relative entropy, $R(\mathcal{Q} \parallel \mathcal{P})$. At this stage we consider distortions to the joint distribution of assets so we impose an overall entropy constraint for all the n assets. Based on Corollary $C3.3$ of Dupuis and Ellis [6], the entropy constraint becomes:

$$R(\mathcal{Q} \parallel \mathcal{P}) = \sum_{i=1}^{n} R(\mathcal{Q}_i \parallel \mathcal{P}_i) = \sum_{i=1}^{n} \int_0^\infty e^{-\delta u} \mathbb{E}_{\mathcal{Q}} \left(\frac{h_i^2}{2} \right) du. \tag{4}$$

The above equation allows us to consider n separate distortion terms, one for each asset. However in order to reduce the complexity of the model, we initially assume symmetric distorted measures $\mathcal{Q}_i$, and examine the case with the same distortion terms h_i. In this specific case, the equation for wealth dynamics becomes:

$$\begin{aligned} dW &= \left(rW - c + \sum_{i=1}^{n} w_i(\alpha_i - r)W + h \sum_{i=1}^{n} \sigma_i R_{ri} w_i W \right) dt \qquad (5) \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n} w_j \sigma_j W R_{ij} d\hat{B}_i \end{aligned}$$

[13] This is the reason for the use of the specific form of equation (1).

where $R_{ri} = \sum_{j=1}^{n} R_{ij}$,[14] is the i^{th} element of the matrix, each of whose elements is equal to the sum of the elements of the ith row of matrix R. Under model misspecification, a multiplier robust control problem can be associated with the problem of maximizing the present value of lifetime expected utility, or:

$$\max_{w_i, C} \mathbb{E}_0 \int_0^{\infty} e^{-\delta t} U(C) dt \tag{6}$$

In this case the multiplier robust control problem becomes:

$$J(\theta) = \sup_{w_i, C} \inf_{h} \mathbb{E}_{\mathcal{Q}} \int_0^{\infty} e^{-\delta t} \big[U(C) + \theta_n \frac{h^2}{2} \big] dt \tag{7}$$

subject to (5).

In the above equation because of (4), $\theta_n = n\theta$ where θ denotes the robustness parameter which takes values greater than or equal to zero. Thus it is assumed that concerns about model misspecification are the same for the price processes of all assets. As shown by Hansen and Sargent [22] θ is the Lagrangian multiplier at the optimum, associated with the entropy constraint $\mathcal{Q}(\tau) = \{Q \in \mathcal{Q} : R_t(\mathcal{Q} \parallel \mathcal{P}) \leq \tau \;\; \forall t\}$. A value of $\theta = \infty$, indicates that we are absolutely sure about the measure $\mathcal{P}$, with no preference for robustness. This case can be regarded as the risk aversion case and the problem is reduced to the standard Merton problem with the objective function given by (6). Lower values for θ indicate preference for robustness under model misspecification, or uncertainty aversion, where a

[14] For two assets we will see (in the next section) that: $R_{r1} = 1$, $R_{r2} = \rho + \sqrt{1 - \rho^2}$, where ρ is the correlation coefficient.

value of $\theta = 0$ indicates that we have no knowledge about the measure $\mathcal{P}$.

Using the results of Fleming and Souganidis [10] regarding the existence of a recursive solution to the multiplier problem, Hansen et al. [24] show that problem (7) can be transformed into a stochastic infinite horizon two-player game between the investor and Nature. Nature plays the role of a "mean agent" and chooses a reduction h in the mean return of assets to reduce the investor's life time utility. The Bellman-Isaacs conditions for this game implies that the value function $V(W, \theta)$ satisfies the following equation:

$$\begin{aligned} \delta V &= \max_{w_i, C} \min_{h} \Big\{ U(C) + \Big(rW - c + \sum_{i=1}^{n} w_i(\alpha_i - r)W \qquad (8) \\ &+ h \sum_{i=1}^{n} \sigma_i R_{ri} w_i W \Big) V_W + \theta_n \frac{h^2}{2} + \frac{1}{2} V_{WW} \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} W^2 \Big\}. \end{aligned}$$

The first-order conditions which describe the solution of the above two-player game are:

$$U'(C) = V_W \qquad (9)$$

$$h = -\frac{V_W W \sum_{i=1}^{n} w_i \sigma_i R_{ri}}{\theta_n} \qquad (10)$$

$$\sum_{j=1}^{n} w_j W \sigma_{ij} = A(\alpha_i - r) + A \sigma_i R_{ri} h \quad , \; i = 1, ..., n \qquad (11)$$

$$A = -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C) \frac{\partial C}{\partial W}} \qquad (12)$$

From the above system of equations, it can be seen that as $\theta \to \infty$ the solution reduces to that of Merton's standard problem given by (2). Using (10) to eliminate h from (11), we obtain the robust portfolio weights, or equivalently, the fraction of the wealth invested in each asset, as:

$$w_i^* W = A \sum_{j=1}^{n} \upsilon_{ij}^{-1} (\alpha_j - r) \ , \ i = 1, ..., n \tag{13}$$

where υ^{-1} is the inverse of the matrix:

$$[\upsilon_{ij}] = [(\Sigma D \Sigma)_{ij}] \tag{14}$$

$$D_{ij} = \big(\rho_{ij} - \frac{V_W^2}{\theta_n V_{WW}} R_{ri} R_{rj}\big) \tag{15}$$

and ρ_{ij} is the correlation coefficient of the benchmark model, between assets i and j ($\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$). In order to determine the change in portfolio weights induced by uncertainty aversion relative to the risk aversion weights, we subtract from (2) the relationship (13) to obtain:[15]

$$W(w_i - w_i^*) = W \Delta w_i = A \sum_{j=1}^{n} [\Sigma^{-1}(\Pi^{-1} - D^{-1})\Sigma^{-1}]_{ij} (\alpha_j - r) \ , \ i = 1, ..., n \tag{16}$$

where $\Pi = RR'$, is the correlation coefficient matrix. If $\Delta w_i \lesseqgtr 0$, then uncertainty aversion, as reflected in robust control portfolio choices, increases (decreases) the holding of asset i as a fraction of wealth invested in this asset, relative to risk aversion.

The change in the total holdings of risky assets as a fraction of wealth between uncertainty and risk aversion is obtained by using (16) as:

$$W \Delta W = W \sum_{i=1}^{n} \Delta w_i = A \sum_{i=1}^{n} \sum_{j=1}^{n} [\Sigma^{-1}(\Pi^{-1} - D^{-1})\Sigma^{-1}]_{ij} (\alpha_j - r) \tag{17}$$

[15] For infinitesimal changes in θ, this is basically a comparative statics exercise that characterizes the derivative $\partial w_i^* / \partial \theta$.

If $\Delta W \lesseqgtr 0$, then uncertainty aversion increases (decreases) the total holdings of risky assets i as a fraction of wealth relative to risk aversion.

To derive conditions under which the signs Δw_i, ΔW can be determined and which are relatively simpler to interpret and present, we consider the special case of two risky assets.

2.1. Two Risky Assets

We consider one risk free asset and two correlated risky assets, where ρ denotes the correlation coefficient at the benchmark model. In this case $B = [B_1, B_2]^T$ is a vector of independent Brownian processes defined on an underlying probability space $(\Omega, \mathcal{F})$, with measure $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$. Because of $\mathbb{E}(dB_1 d\hat{B}_2) = \rho dt$, where $\mathbb{E}$ denotes expected value and $dB_1, d\hat{B}_2$ are correlated Brownian motions on $\mathcal{P}_1, \mathcal{P}_2$ respectively, the evolution of the prices of the assets based on (1) is given by:[16]

$$dP_1(t) = \alpha_1 P_1(t)dt + \sigma_1 P_1(t)dB_1(t) \qquad t \geq 0 \tag{18}$$

$$dP_2(t) = \alpha_2 P_2(t)dt + \sigma_2 P_2(t)\rho dB_1(t) + \sigma_2 P_2(t)\sqrt{1-\rho^2}dB_2(t) \tag{19}$$

Merton's solution for the maximization problem (6) in the two risky

[16] We have that for independent Brownian motions B_1, B_2: $\mathbb{E}(dB_1 dB_2) = 0$, $\mathbb{E}(dB_1 dB_1) = dt$ so we write $d\hat{B}_2 = \rho dB_1 + \sqrt{1-\rho^2}dB_2$.

assets case is:

$$
\begin{aligned}
w_1 W &= \frac{A(\alpha_1 - r)\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} - \frac{A(\alpha_2 - r)\sigma_{12}}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} && (20)\\
w_2 W &= -\frac{A(\alpha_1 - r)\sigma_{12}}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} + \frac{A(\alpha_2 - r)\sigma_2^2}{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)} && (21)\\
A &= -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C)\frac{\partial C}{\partial W}} && (22)
\end{aligned}
$$

If we perturb each Brownian motion separately, the wealth equation becomes:

$$
\begin{aligned}
dW &= w_1(\alpha_1 - r + \sigma_1 h)W dt + w_2\big(\alpha_2 - r + \sigma_2(\rho h + h\sqrt{1-\rho^2})\big)W dt \\
&\quad +(rW - C)dt + W\sigma_1 w_1 d\hat{B}_1 + W\sigma_2\rho w_2 d\hat{B}_1 + \\
&\quad \sigma_2\sqrt{1-\rho^2}w_2 d\hat{B}_2. && (23)
\end{aligned}
$$

In this specific case the Bellman equation for problem (7) subject to (23) is:

$$
\begin{aligned}
\delta V &= \max_{w_i, C}\min_{h}\Big\{U(C) + \Big(w_1(\alpha_1 - r + \sigma_1 h)W + (rW - c) + \theta_2\frac{h^2}{2} + \\
&\quad w_2(\alpha_2 - r + \sigma_2(\rho + \sqrt{1-\rho^2})h)W\Big)V_W + \\
&\quad \frac{1}{2}V_{WW}\sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j \sigma_{ij} W^2\Big\}, && (24)
\end{aligned}
$$

where $\theta_2 = 2\theta$ and θ this time refers to the robustness parameter in the two assets case. The first order conditions which describe the solution of the above two-player game are:

$$
\begin{aligned}
U'(C) &= V_W & (25)\\
h &= -\frac{V_W W\big(\sigma_1 w_1^* + \sigma_2(\rho+\sqrt{1-\rho^2})w_2^*\big)}{\theta_2} & (26)\\
\sum_{j=1}^{2} w_j^* W \sigma_{1j} &= A(\alpha_1 - r) + A\sigma_1 h & (27)\\
\sum_{j=1}^{2} w_j^* W \sigma_{2j} &= A(\alpha_2 - r) + A\sigma_2(\rho+\sqrt{1-\rho^2})h & (28)\\
A &= -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C)\frac{\partial C}{\partial W}}. & (29)
\end{aligned}
$$

Using matrix notation the solution of the above problem can be described by the following equation:

$$
\left[\begin{array}{cc} w_1^* W & w_2^* W \end{array}\right] \Lambda = \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r) \end{array}\right] \tag{30}
$$

where:

$$
\Lambda = \left[\begin{array}{cc} \sigma_{11}\big(1 - \frac{V_W^2}{\theta_2 V_{WW}}\big) & \sigma_{12}\big(1 - \frac{V_W^2}{\theta_2 V_{WW}}\frac{\rho+\sqrt{1-\rho^2}}{\rho}\big) \\ \sigma_{12}\big(1 - \frac{V_W^2}{\theta_2 V_{WW}}\frac{\rho+\sqrt{1-\rho^2}}{\rho}\big) & \sigma_{22}\big(1 - \frac{V_W^2}{\theta_2 V_{WW}}(1+2\rho\sqrt{1-\rho^2})\big) \end{array}\right]. \tag{31}
$$

If Σ denotes the diagonal matrix with elements σ_1, σ_2 then:

$$
\Lambda = \Sigma \left[\begin{array}{cc} \big(1 - \frac{V_W^2}{\theta_2 V_{WW}}\big) & \big(\rho - \frac{V_W^2}{\theta_2 V_{WW}}(\rho+\sqrt{1-\rho^2})\big) \\ \big(\rho - \frac{V_W^2}{\theta_2 V_{WW}}(\rho+\sqrt{1-\rho^2})\big) & \big(1 - \frac{V_W^2}{\theta_2 V_{WW}}(1+2\rho\sqrt{1-\rho^2})\big) \end{array}\right] \Sigma. \tag{32}
$$

Solving the above system we determine the fraction of the wealth invested in the first and second asset under robust portfolio choices as:

$$\left[\begin{array}{cc} w_1^* W & w_2^* W \end{array}\right] = \frac{1}{(1-\rho^2)\left(1-2\frac{V_W^2}{\theta_2 V_{WW}}\right)} \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r) \end{array}\right]$$
$$\Sigma^{-1} \left[\begin{array}{cc} \left(1-\frac{V_W^2}{\theta_2 V_{WW}}(1+2\rho\sqrt{1-\rho^2}\right) & -\rho+\frac{V_W^2}{\theta_2 V_{WW}}(\rho+\sqrt{1-\rho^2}) \\ -\rho+\frac{V_W^2}{\theta_2 V_{WW}}(\rho+\sqrt{1-\rho^2}) & (1-\frac{V_W^2}{\theta_2 V_{WW}}) \end{array}\right] \Sigma^{-1} \quad (33)$$

Next we examine, as in the previous section, the changes in the robust portfolio weights $\Delta w_i = w_i - w_i^*,\ \ i = 1, 2$ between risk aversion $(\theta \to \infty)$ and uncertainty aversion $(\theta < \infty)$. Using (20),(21),(33) we obtain:

$$\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = \frac{1}{(1-\rho^2)} \left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r) \end{array}\right] \Sigma^{-1} \Xi \Sigma^{-1} \quad (34)$$

where:

$$\Xi = \left[\begin{array}{cc} 1 & -\rho \\ -\rho & 1 \end{array}\right] - \frac{1}{\left(1-2\frac{V_W^2}{\theta_2 V_{WW}}\right)} \quad (35)$$
$$\left[\begin{array}{cc} \left(1-\frac{V_W^2}{\theta_2 V_{WW}}(1+2\rho\sqrt{1-\rho^2}\right) & -\rho+\frac{V_W^2}{\theta_2 V_{WW}}(\rho+\sqrt{1-\rho^2}) \\ -\rho+\frac{V_W^2}{\theta_2 V_{WW}}(\rho+\sqrt{1-\rho^2}) & (1-\frac{V_W^2}{\theta_2 V_{WW}}) \end{array}\right]$$

After some manipulations we have that:

$$\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = \quad (36)$$
$$\frac{AV_W^2 \left[\begin{array}{cc} \alpha_1 - r & \alpha_2 - r \end{array}\right]}{(1-\rho^2)(\theta_2 V_{WW} - V_W^2)} \Sigma^{-1} \left[\begin{array}{cc} 2\rho\sqrt{1-\rho^2}-1 & \rho-\sqrt{1-\rho^2} \\ \rho-\sqrt{1-\rho^2} & -1 \end{array}\right] \Sigma^{-1}.$$

Therefore the solution becomes:

$$
\begin{aligned}
W\Delta w_1 &= \frac{\kappa}{\sigma_1}\left[\frac{\alpha_1 - r}{\sigma_1}(2\rho\sqrt{1-\rho^2} - 1) + \frac{\alpha_2 - r}{\sigma_2}(\rho - \sqrt{1-\rho^2})\right] &(37)\\
W\Delta w_2 &= \frac{\kappa}{\sigma_2}\left[\frac{\alpha_1 - r}{\sigma_1}(\rho - \sqrt{1-\rho^2}) - \frac{\alpha_2 - r}{\sigma_2}\right] &(38)\\
\kappa &= \frac{AV_W^2}{(1-\rho^2)(\theta_2 V_{WW} - V_W^2)}. &(39)
\end{aligned}
$$

In the above equation, κ is always a negative number, so denoting the relative price of risk by:

$$\lambda = \frac{\frac{\alpha_2 - r}{\sigma_2}}{\frac{\alpha_1 - r}{\sigma_1}} \tag{40}$$

we obtain

$$
\begin{aligned}
W\Delta w_1 &< 0 \quad \text{if} \quad \frac{\rho - \sqrt{1-\rho^2}}{1 - 2\rho\sqrt{1-\rho^2}} > \frac{1}{\lambda} &(41)\\
W\Delta w_2 &< 0 \quad \text{if} \quad \rho - \sqrt{1-\rho^2} > \lambda. &(42)
\end{aligned}
$$

It can be seen that independent of the specific form of utility function and the value of the robustness parameter θ, the fraction of the wealth invested in the first asset increases relative to Merton's weight if $\frac{\rho - \sqrt{1-\rho^2}}{1-2\rho\sqrt{1-\rho^2}} > \frac{1}{\lambda}$, while the fraction of the wealth invested in the second asset increases relative to Merton's weight if $\rho - \sqrt{1-\rho^2} > \lambda$. If we combine (42) and (41), it can be seen that it is not possible to have:

$$\frac{1 - 2\rho\sqrt{1-\rho^2}}{\rho - \sqrt{1-\rho^2}} < \lambda < \rho - \sqrt{1-\rho^2}. \tag{43}$$

So both weights cannot increase at the same time due to uncertainty aversion. If the fraction of the wealth invested in the first asset increases relative to Merton's weight, then the fraction of the wealth invested in the second asset has to decrease relative to Merton's weight. Furthermore, the holdings of both assets decrease relative to Merton's weights if $\rho < \sqrt{2}/2$, thus uncertainty aversion results in an increase in the holdings of one asset for sufficiently high and positive correlation between the two risky assets.

The effect of uncertainty aversion on the total holdings of both risky assets is obtained by combining (41) and (42) as:

$$
\begin{aligned}
W\Delta W = W\Delta w_1 + W\Delta w_2 = \qquad (44)\\
\kappa\frac{\alpha_1 - r}{\sigma_1}\frac{1}{\sigma_2}\big[(2\rho\sqrt{1-\rho^2}-1)\sigma + \lambda\sigma(\rho - \sqrt{1-\rho^2}) +\\
(\rho - \sqrt{1-\rho^2}) - \lambda\big] < 0 \quad if\\
(\lambda\sigma + 1)(\rho - \sqrt{1-\rho^2}) > \lambda + \sigma(1 - 2\rho\sqrt{1-\rho^2})\\
\hat{\lambda}(\rho - \sqrt{1-\rho^2} - \frac{1}{\sigma}) > \sigma(1 - 2\rho\sqrt{1-\rho^2}) - (\rho - \sqrt{1-\rho^2}) \qquad (45)\\
\text{with } \hat{\lambda} = \frac{a_2 - r}{a_1 - r},\ \sigma = \frac{\sigma_2}{\sigma_1}
\end{aligned}
$$

If (45) is satisfied then asset holdings increase under uncertainty aversion.

These results can be summarized in the following proposition:

PROPOSITION 1. *With λ, $\hat{\lambda}$, σ, and ρ as defined above, robust portfolio choices under uncertainty aversion imply, for a market consisting of one riskless and two risky assets, the following:*

1. *If $\hat{\lambda} > \frac{\sigma^2(1-2\rho\sqrt{1-\rho^2})-\sigma(\rho-\sqrt{1-\rho^2})}{\sigma\left(\rho-\sqrt{1-\rho^2}\right)-1}$, there is an increase in the holdings of risky assets, relative to risk aversion, or $\Delta W < 0$.*

2. *If $\lambda > \frac{1-2\rho\sqrt{1-\rho^2}}{\rho-\sqrt{1-\rho^2}}$, there is an increase in the holdings of the first*

risky asset, relative to risk aversion, or $\Delta w_1 < 0$.

3. *If* $\lambda < \rho - \sqrt{1-\rho^2}$, *there is an increase in the holdings of the second risky asset, relative to risk aversion, or* $\Delta w_2 < 0$.

4. *An increase in the holdings of one risky asset under robust portfolio choices implies a reduction in the holdings of the other risky asset.*

5. *When concerns about model misspecification do not exist, or* $\theta \to \infty$, *then the difference in portfolio choices between uncertainty aversion and risk aversion vanish.* $\Delta W \to 0$, $(\Delta w_1, \Delta w_2) \to 0$.

6. If $\rho < \sqrt{2}/2$, the holdings of both assets decrease relative to Merton's weights.

Figures 1 and 2 provide a graphical exposition of these results. In figure 1 for parameter constellations such that the value of $\hat{\lambda}$ is above the surface $\phi(\sigma, \rho) = \frac{\sigma^2(1-2\rho\sqrt{1-\rho^2}) - \sigma(\rho - \sqrt{1-\rho^2})}{\sigma\left(\rho - \sqrt{1-\rho^2}\right) - 1}$, robust control implies an increase in asset holdings. In figure 2 for parameter constellations such that the value of λ is above the line $\psi(\rho) = \frac{1-2\rho\sqrt{1-\rho^2}}{\rho - \sqrt{1-\rho^2}}$, robust control implies an increase in the holdings of asset one and a decrease in the holdings of asset two.

[Figure 1]

[Figure 2]

3. DIFFERENCES IN PREFERENCES FOR ROBUSTNESS

In this section we generalize our model to allow for differences in preferences for robustness, or differences in concern about model misspecification among the different assets. These differences can also be interpreted as differences in the levels of ambiguity associated with the price processes of each asset.[17]

Following the robust control methodology, we solve the same problem as in section 2, placing n different penalty terms h_i , $i = 1, ..., n$ and considering n different robustness parameters θ_i, one for each asset. Our results regarding changes in asset holdings between uncertainty and risk aversion in the generalized model are summarized in the following proposition.

PROPOSITION 2. *Robust portfolio choices under uncertainty aversion imply, for a market consisting of one riskless and multiple risky assets with differences in ambiguity among assets, the following:*

1. The change in the fraction of wealth invested in each asset, between uncertainty and risk aversion, is:
   $W\Delta w_i = \sum_{j=1}^{n}[\Sigma^{-1}(\Pi^{-1} - \hat{D}^{-1})\Sigma^{-1}]_{ij}(\alpha_j - r)$, $i = 1, ..., n$.

2. The change in the total holdings of risky assets as a fraction of wealth between uncertainty and risk aversion is:

[17] Uppal and Wang [31] develop a framework that allows for ambiguity about the joint distribution for all stocks being considered for the portfolio and also for different levels of ambiguity for the marginal distribution for any subset of the stocks.

$W\Delta W = \sum_{i=1}^{n} \Delta w_i = \sum_{i=1}^{n}\sum_{j=1}^{n}[\Sigma^{-1}(\Pi^{-1} - \hat{D}^{-1})\Sigma^{-1}]_{ij}(\alpha_j - r),$

where $\hat{D}_{ij} = \left(\rho_{ij} - \frac{V_W^2}{V_{WW}}\sum_{\kappa=1}^{n}\frac{R_{i\kappa}R_{jk}}{\theta_\kappa}\right).$

For the proof see Appendix.

If we compare the above result with the corresponding one derived in section 2, we see that the only difference is that matrix D has been replaced by matrix $\hat{D}$, which incorporates the heterogeneity in robustness parameters.

3.1. The Special Case of Two Risky Assets

In this subsection we examine the case of two risky assets, with different penalty terms, h_1, h_2, and robustness parameters, θ_1, θ_2. Following the proof of the previous section, we obtain the following proposition:

PROPOSITION 3. *Robust portfolio choices under uncertainty aversion imply, for a market consisting of one riskless and two risky assets with differences in ambiguity among the two assets, the following:*

1. $W\Delta w_1 = \frac{\kappa}{\sigma_1}\left[\frac{\alpha_1 - r}{\sigma_1}\left(\frac{\theta_1-\theta_2}{\theta_2}\frac{1-\rho^2}{1-\frac{V_W^2}{V_{WW}}} - 1\right) + \frac{\alpha_2 - r}{\sigma_2}\rho\right] < 0$, if $\frac{\theta_1-\theta_2}{\theta_2}\mu > \frac{1-\rho\lambda}{1-\rho^2}$

   $W\Delta w_2 = \frac{\kappa}{\sigma_2}\left[\frac{\alpha_1 - r}{\sigma_1}\rho - \frac{\alpha_2 - r}{\sigma_2}\right] < 0$, if $\rho > \lambda$

2. *We never increase the holdings of both assets at the same time, relative to risk aversion.*

3. $W\Delta W < 0$, *if* $\rho\lambda\sigma + \rho - \lambda = \hat{\lambda}\left(\rho - \frac{1}{\sigma}\right) + \rho > \sigma\left[1 - \frac{\theta_1-\theta_2}{\theta_2}\mu(1-\rho^2)\right]$

where $\mu = \frac{1}{1-\frac{V_W^2}{V_{WW}}}$, $\quad \sigma = \frac{\sigma_2}{\sigma_1}$, $\lambda = \frac{\frac{\alpha_2 - r}{\sigma_2}}{\frac{\alpha_1 - r}{\sigma_1}}$, $\hat{\lambda} = \frac{a_2 - r}{a_1 - r}$

For the proof see Appendix.

The first condition of this proposition implies that if preferences for robustness for asset one are low (high θ_1) or the ambiguity associated with the price process for this asset is small, while preferences for robustness for asset two are high (low θ_2) or the ambiguity associated with the price process for this asset is large, then it very likely that the holdings of the first asset will increase under uncertainty aversion. If asset one is a home asset and asset two is a foreign asset, then this result provides some explanation for the home bias puzzle. Uppal and Wang [31] derive a similar result regarding the home puzzle bias. They, however, use the normalization that essentially endogenizes the robustness parameter and then breaks down the consistency of preferences with the Gilboa and Schmeidler axiomatization of uncertainty aversion. Pathak [26] also provides an explanation of the home bias puzzle using a two-asset model and a $\kappa - Ignorance$ framework, where the worst-case scenario is used to reduce the mean return of the asset price process. There is a subtle difference between our result and the $\kappa - Ignorance$, worst case scenario approach. In the latter approach the worst case scenario means that the reduction in the mean return of the asset price process is determined at the level where the entropy constraint $\mathcal{Q}(\tau) = \{Q \in \mathcal{Q} : R_t(\mathcal{Q} \parallel \mathcal{P}) \leq \tau \;\; \forall t\}$ is binding. In the robust control model developed in this paper, the robustness parameter associated with the penalty terms is the Lagrangian multiplier associated with the entropy constraint. In the two-player game between the investor and Nature (the mean agent) described by (8), the choice of the penalty term **h** by Nature, which reduces the mean return, is not necessarily set at the constraint constants of the entropy constraints but is chosen in a "penalty maximizing way" as shown in (50) of the Appendix in the proof of proposition 2. Thus our result may be interpreted as an additional explanation of the home bias puzzle using a different angle for incorporating uncertainty aversion. Furthermore since the holdings in each asset depend on the θs, this approach could associate the magnitude of the puzzle with differences in uncertainty aversion between home and foreign assets.

4. CONCLUDING REMARKS

Robust portfolio rules suggest that total holdings of risky assets may increase under uncertainty aversion relative to the risk aversion case, which is a result that can be contrasted to results suggesting that robust methods in portfolio selection imply a reduction in the total holdings of risky assets. Furthermore under heterogeneity with respect to preference for robustness, robust portfolio rules suggest that the investor might increase the holdings of the less ambiguous asset and reduce the holdings of the more ambiguous asset, a result that might provide an additional explanation for the home bias puzzle.

The robust portfolio rules derived in this paper are parametrized by the robustness parameter θ, which is not endogenized in order to keep the model consistent with the Gilboa Schmeilder axiomatization of uncertainty aversion.[18] Thus our results should be regarded mainly as comparative static results indicating the direction of changes in risky asset holdings when preferences for robustness changes, with the limiting case being the no preference for robustness, which is equivalent to standard risk aversion. The fact that changes could go either way depending on the structure of the model parameters suggests that uncertainty aversion and adoption of robust portfolio rules should not be associated with smaller holdings of risky assets.

[18] Thus the full characterization of the robust portfolio rule requires estimation of θ, using for example detection probabilities [23].

Appendix

Proof of Proposition 2

Suppose now, that we place n different penalty terms h_i for $i = 1, ..., n$ and consider the corresponding robustness parameters θ_i. In this case, the equation for wealth dynamics takes the form:

$$\begin{aligned} dW &= \Big(rW - c + \sum_{i=1}^{n} w_i(\alpha_i - r)W + \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i R_{ij} w_i W h_j\Big)dt \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} w_j \sigma_j W R_{ij} d\hat{B}_i \end{aligned} \tag{46}$$

and the corresponding multiplier robust control problem now becomes:

$$J(\theta) = \sup_{w_i, C} \inf_{h_i} \mathbb{E}_{\mathcal{Q}} \int_0^{\infty} e^{-\delta t} \big[U(C) + \sum_{i=1}^{n} \theta_i \frac{h_i^2}{2}\big] dt \tag{47}$$

subject to (46).

The value function $V(W, \theta)$ satisfies the following equation:

$$\begin{aligned} \delta V &= \max_{w_i, C} \min_{h_i} \Big\{ U(C) + \Big(rW - c + \sum_{i=1}^{n} w_i(\alpha_i - r)W \qquad (48) \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i R_{ij} w_i W h_j\Big) V_W + \sum_{i=1}^{n} \theta_i \frac{h_i^2}{2} + \frac{1}{2} V_{WW} \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} W^2 \Big\}. \end{aligned}$$

The first-order conditions are:

$$U'(C) = V_W \tag{49}$$

$$h_i = -\frac{V_W W \sum_{j=1}^{n} w_j \sigma_j R_{ji}}{\theta_i}, \ i = 1, ..., n \tag{50}$$

$$\sum_{j=1}^{n} w_j W \sigma_{ij} = A(\alpha_i - r) + A\sigma_i \sum_{j=1}^{n} h_j R_{ij}, \ i = 1, ..., n \tag{51}$$

$$A = -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C)\frac{\partial C}{\partial W}}. \tag{52}$$

Replacing the second into the third relationship we obtain that:

$$w_i^* W = \sum_{j=1}^{n} \hat{\upsilon}_{ij}^{-1} (\alpha_j - r) \ , \ i = 1, ..., n \tag{53}$$

where $\hat{\upsilon}^{-1}$ is the inverse of the matrix:

$$[\hat{\upsilon}_{ij}] = [(\Sigma \hat{D} \Sigma)_{ij}] \tag{54}$$

$$\hat{D}_{ij} = \big(\rho_{ij} - \frac{V_W^2}{V_{WW}} \sum_{\kappa=1}^{n} \frac{R_{i\kappa} R_{jk}}{\theta_\kappa}\big). \tag{55}$$

So in order to compare again the change of wealth invested in each one of the assets, we subtract relationship (53) from (2). Therefore we obtain that:

$$W \Delta w_i = \sum_{j=1}^{n} [\Sigma^{-1} (\Pi^{-1} - \hat{D}^{-1}) \Sigma^{-1}]_{ij} (\alpha_j - r) \ , \ i = 1, ..., n \tag{56}$$

From the previous equation we obtain:

$$W \Delta W = \sum_{i=1}^{n} \Delta w_i = \sum_{i=1}^{n} \sum_{j=1}^{n} [\Sigma^{-1} (\Pi^{-1} - \hat{D}^{-1}) \Sigma^{-1}]_{ij} (\alpha_j - r). \tag{57}$$

■

Proof of Proposition 3

In this case the equation for wealth dynamics becomes:

$$\begin{aligned} dW &= \Big[w_1(\alpha_1 - r + \sigma_1 h_1) + w_2 \big(\alpha_2 - r + \sigma_2 (\rho h_1 + h_2 \sqrt{1 - \rho^2}) \big) \Big] W dt \\ &\quad + (rW - C) dt + W \sigma_1 w_1 d\hat{B}_1 + W \sigma_2 \rho w_2 d\hat{B}_1 \\ &\quad + \sigma_2 \sqrt{1 - \rho^2} w_2 d\hat{B}_2. \end{aligned} \tag{58}$$

The multiplier robust control problem is defined as:

$$J(\theta) = \sup_{w_i, C} \inf_{h} \mathbb{E}_{\mathcal{Q}} \int_0^\infty e^{-\delta t} \big[U(C) + \theta_1 \frac{h_1^2}{2} + \theta_2 \frac{h_2^2}{2} \big] dt \tag{59}$$

subject to (58).

The value function in this case satisfies:

$$\begin{aligned} \delta V &= \max_{w_i, C} \min_{h_i} \Big\{ U(C) + \Big(w_1(\alpha_1 - r + \sigma_1 h_1)W + (rW - c) \\ &\quad + \theta_1 \frac{h_1^2}{2} + \theta_2 \frac{h_2^2}{2} + w_2(\alpha_2 - r + \sigma_2 \rho h_1 + \sigma_2 \sqrt{1-\rho^2} h_2)W \Big) V_W \\ &\quad + \frac{1}{2} V_{WW} \sum_{i=1}^{2} \sum_{j=1}^{2} w_i w_j \sigma_{ij} W^2 \Big\}. \end{aligned} \tag{60}$$

For this two-player game the first-order conditions are:

$$U'(C) = V_W \tag{61}$$

$$h_1 = -\frac{V_W W \big(\sigma_1 w_1^* + \sigma_2 \rho w_2^*\big)}{\theta_1} \tag{62}$$

$$h_2 = -\frac{V_W W \sqrt{1-\rho^2} \sigma_2 w_2^*}{\theta_2} \tag{63}$$

$$\sum_{j=1}^{2} w_j^* W \sigma_{1j} = A(\alpha_1 - r) + A \sigma_1 h_1 \tag{64}$$

$$\sum_{j=1}^{2} w_j^* W \sigma_{2j} = A(\alpha_2 - r) + A \sigma_2 (\rho h_1 + \sqrt{1-\rho^2} h_2) \tag{65}$$

$$A = -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C)\frac{\partial C}{\partial W}}. \tag{66}$$

Using matrices as in section 3, we are able to describe the solution of the above problem by the following equation:

$$\begin{bmatrix} w_1^* W & w_2^* W \end{bmatrix} \Lambda = \begin{bmatrix} A(\alpha_1 - r) & A(\alpha_2 - r) \end{bmatrix} \tag{67}$$

where now Λ is the matrix:

$$\Lambda = \begin{bmatrix} \sigma_{11}(1 - \frac{V_W^2}{\theta_1 V_{WW}}) & \sigma_{12}(1 - \frac{V_W^2}{\theta_1 V_{WW}}) \\ \sigma_{21}(1 - \frac{V_W^2}{\theta_1 V_{WW}}) & \sigma_{22}\big(1 - \frac{V_W^2}{V_{WW}}(\frac{\rho^2}{\theta_1} + \frac{1-\rho^2}{\theta_2})\big) \end{bmatrix}. \tag{68}$$

In this case the solution is:

$$\left[\begin{array}{cc} w_1^* W & w_2^* W \end{array}\right] = \frac{\left[\begin{array}{cc} A(\alpha_1 - r) & A(\alpha_2 - r) \end{array}\right]}{(1-\rho^2)(1-\frac{V_W^2}{\theta_1 V_{WW}})(1-\frac{V_W^2}{\theta_2 V_{WW}})} \tag{69}$$

$$\Sigma^{-1}\left[\begin{array}{cc} \left(1-\frac{V_W^2}{V_{WW}}(\frac{\rho^2}{\theta_1}+\frac{1-\rho^2}{\theta_2})\right) & \rho - \rho\frac{V_W^2}{\theta_1 V_{WW}} \\ \rho - \rho\frac{V_W^2}{\theta_1 V_{WW}} & 1-\frac{V_W^2}{\theta_1 V_{WW}} \end{array}\right]\Sigma^{-1}.$$

We examine the difference between the quantities given by (20),(21) and (69) and after manipulations we obtain:

$$\left[\begin{array}{cc} W\Delta w_1 & W\Delta w_2 \end{array}\right] = \frac{AV_W^2\left[\begin{array}{cc} \alpha_1 - r & \alpha_2 - r \end{array}\right]}{(1-\rho^2)(\theta_2 V_{WW} - V_W^2)} \tag{70}$$

$$\Sigma^{-1}\left[\begin{array}{cc} \frac{\theta_1-\theta_2}{\theta_2}\frac{1-\rho^2}{1-\frac{V_W^2}{V_{WW}}} - 1 & \rho \\ \rho & -1 \end{array}\right]\Sigma^{-1}.$$

Therefore the solution becomes:

$$W\Delta w_1 = \frac{\kappa}{\sigma_1}\left[\frac{\alpha_1 - r}{\sigma_1}(\frac{\theta_1-\theta_2}{\theta_2}\frac{1-\rho^2}{1-\frac{V_W^2}{V_{WW}}} - 1) + \frac{\alpha_2 - r}{\sigma_2}\rho\right] \tag{71}$$

$$W\Delta w_2 = \frac{\kappa}{\sigma_2}\left[\frac{\alpha_1 - r}{\sigma_1}\rho - \frac{\alpha_2 - r}{\sigma_2}\right] \tag{72}$$

$$\kappa = \frac{AV_W^2}{(1-\rho^2)(\theta_2 V_{WW} - V_W^2)}. \tag{73}$$

Thus if λ as in (40) then:

$$W\Delta w_1 < 0 \quad if \quad \frac{\theta_1 - \theta_2}{\theta_2}\mu > \frac{1-\rho\lambda}{1-\rho^2} \tag{74}$$

$$W\Delta w_2 < 0 \quad if \quad \rho > \lambda \tag{75}$$

where $\mu = \frac{1}{1-\frac{V_W^2}{V_{WW}}}$. If equation (75) holds then the right hand side of (74) is always greater than one and the left less than one. So as in the case with the same levels of ambiguity, we never increase the holdings of both assets at the same time. In addition if we combine (71) – (72), we obtain:

$$
\begin{aligned}
W\Delta W = W\Delta w_1 + W\Delta w_2 &= \qquad (76)\\
\kappa\frac{\alpha_1 - r}{\sigma_1}\frac{1}{\sigma_2}\big[\sigma\frac{\theta_1-\theta_2}{\theta_2}\mu(1-\rho^2) - \sigma &+ \rho\lambda\sigma + \rho - \lambda\big] < 0 \quad if \qquad (77)\\
\rho\lambda\sigma + \rho - \lambda &> \sigma\big[1 - \frac{\theta_1-\theta_2}{\theta_2}\mu(1-\rho^2)\big] \text{ or}\\
\hat{\lambda}(\rho - \frac{1}{\sigma}) + \rho &> \sigma\big[1 - \frac{\theta_1-\theta_2}{\theta_2}\mu(1-\rho^2)\big] \qquad (78)
\end{aligned}
$$

where again $\sigma = \frac{\sigma_2}{\sigma_1}$ and $\hat{\lambda} = \frac{a_2 - r}{a_1 - r}$. ■

REFERENCES

[1] W. Brainard, Uncertainty and the effectiveness of policy, Amer. Econom. Rev. 58 (1967) 411-425.

[2] W.A. Brock, S. Durlauf, Elements of a theory of design limits to optimal policy, University of Wisconsin, SSRI Working Paper 2003-25, 2003.

[3] W.A. Brock, S. Durlauf, K. West, Policy evaluation in uncertain economic environments, University of Wisconsin, SSRI Working Paper 2003-15, 2003.

[4] W. A. Brock, A. Xepapadeas, Regulating nonlinear environmental Systems under Knightian uncertainty, in: R. Arnott, B. Greenwald, R. Kanbur, B. Nalebuff (Eds.), Economics for an Imperfect World: Essays In Honor of Joseph Stiglitz, MIT Press, Cambridge, Mass., 2003.

[5] J. Dow, S. Werlang, Uncertainty aversion, risk aversion, and the optimal choice of portfolio, Econometrica 60 (1992) 197-204.

[6] P. Dupuis, R. Ellis, A weak convergence approach to the theory of large deviations, Wiley Series in Probability and Statistics, John Wiley and Sons, New York, 1997.

[7] Z. Chen, L. Epstein, Ambiguity risk and asset returns in continuous time, Econometrica 70 (2002) 1403-1443.

[8] L. Epstein, J. Miao, A two-person dynamic equilibrium under ambiguity, J. Econom. Dynamics and Control, 27 (2003) 1253-1288.

[9] L. Epstein, T. Wang, Intertemporal asset pricing under Knightian uncertainty, Econometrica 63 (1994) 283-322.

[10] W.H. Fleming, P.E. Souganidis, On the existence of value function of two-player, zero sum stochastic differential games, Indiana Univ. Mathematics J. 3 (1989) 293-314.

[11] K.R. French, J. M. Poterba, Investor diversification and international equity markets, Amer. Econom. Rev. Papers and Proceedings 81(2) (May 1991) 222-226.

[12] I. Gilboa, D. Schmeidler, Maxmin expected utility with non-unique prior, J. Math. Econom. 18 (1989) 141-153.

[13] W-F. Liu, Savings and portfolio decisions: Observable implications of Knightian uncertainty, Working Paper, Department of Economics, University of Washington, 2003.

[14] W-F. Liu, Heterogeneous agents economies with Knightian uncertainty, Working Paper, Department of Economics, University of Washington, 2003.

[15] R. Merton, Optimum consumption and portfolio rules in a continuous time model, J. Econom. Theory 3 (1971) 373-413.

[16] R. Merton, An intertemporal capital asset pricing model, Econometrica 41 (1973) 867-887.

[17] A. Onatski, J. Stock, Robust monetary policy under model uncertainty in a small model of the U.S. economy, Macroeconomic Dynamics 6 (2002) 85-110.

[18] A. Onatski, N. Williams, Modeling model uncertainty, NBER Working Paper 9566, 2003.

[19] L. Hansen, T. Sargent, Robust control and filtering for macroeconomics, unpublished manuscript, University of Chicago and Stanford University, 2001.

[20] L. Hansen, T. Sargent, Robust control and model uncertainty, Amer. Econom. Rev. 91 (2001) 60–66.

[21] L. Hansen, T. Sargent, Wanting robustness in macroeconomics, unpublished paper, University of Chicago and Stanford University, 2001.

[22] L. Hansen, T. Sargent, Robust control and economic model uncertainty, unpublished paper, University of Chicago and Stanford University, 2002.

[23] L. Hansen, T. Sargent, Misspecification in recursive macroeconomic theory, monograph, University of Chicago and Stanford University, 2003.

[24] L. Hansen, T. Sargent, G. Turmuhambetova, N. Williams, Robust control and model uncertainty, unpublished paper, University of Chicago and Stanford University, 2002.

[25] P. Maenhout, Robust portfolio rules, hedging and asset pricing, unpublished paper, INSEAD, 2001.

[26] P. Pathak, Notes on robust portfolio choice, unpublished paper, Harvard University, 2002.

[27] C. Roseta-Palma, A. Xepapadeas, Robust control in water management, J. Risk and Uncertainty, forthcoming.

[28] A. Serrat, A dynamic equilibrium model of international portfolio holdings, Econometrica 69(6) (2001) 1467-1490.

[29] U. Soderstrom, Monetary policy with uncertain parameters, Scandinavian J. Econom. 104 (2002) 125-145.

[30] N. Strong, X. Xu, Understanding the equity home bias: Evidence from survey data, Rev. Econ. Statist. 85(2) (2003) 307 - 312.

[31] R. Uppal, T. Wang, Model misspecification and underdiversification, J. Finance, forthcoming.

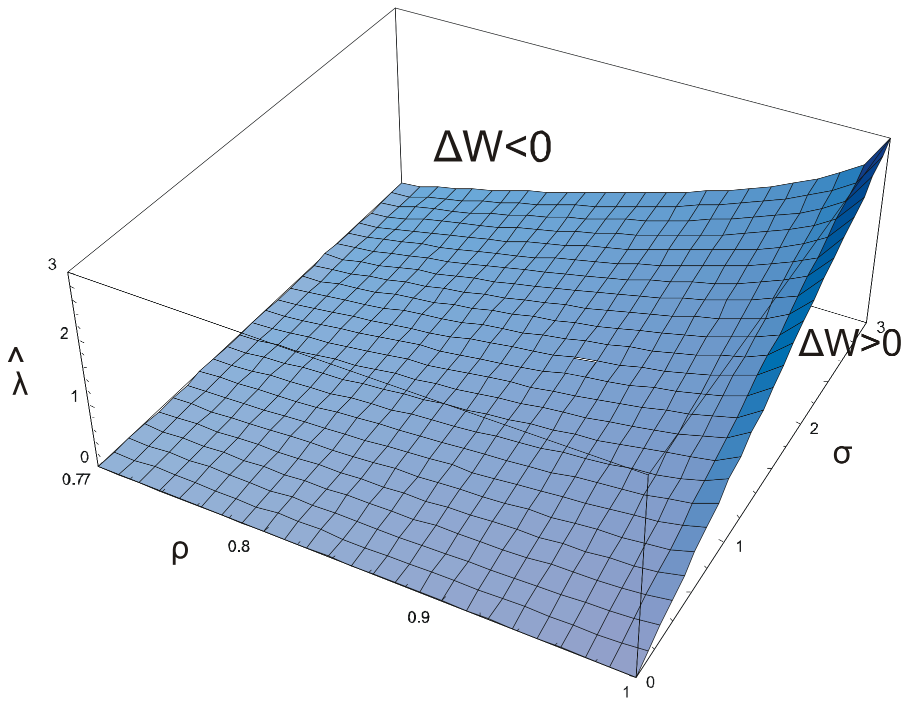

FIG. 1 Changes in the total holdings of risky assets under uncertainty aversion

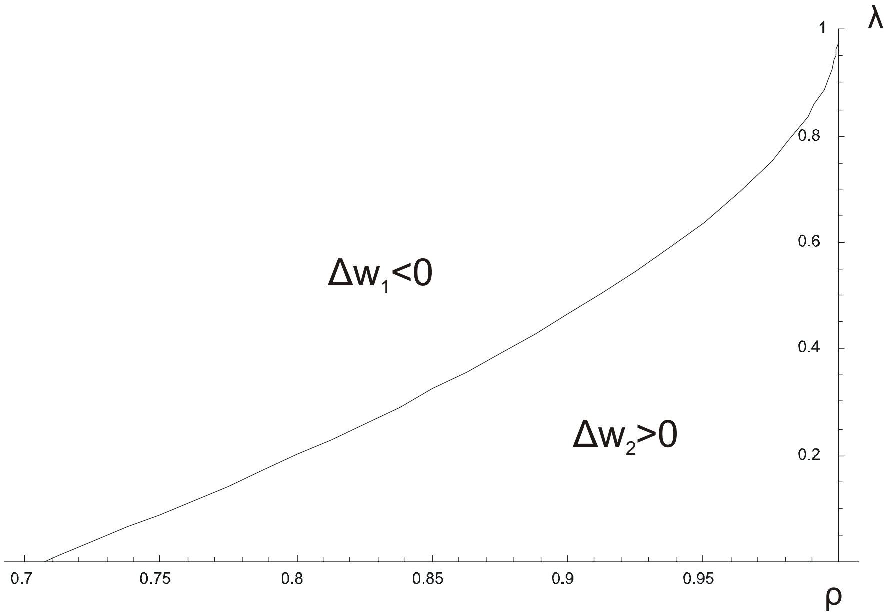

FIG. 2 Changes in the holdings of each risky asset under uncertainty aversion