Spatial Analysis: Development of Descriptive and Normative Methods with Applications to Economic-Ecological Modelling

Abstract

This paper adapts Turing analysis and applies it to dynamic bioeconomic problems where the interaction of coupled economic and ecological dynamics over space endogenously creates (or destroys) spatial heterogeneity. It also extends Turing analysis to standard recursive optimal control frameworks in economic analysis and applies it to dynamic bioeconomic problems where the interaction of coupled economic and ecological dynamics under optimal control over space creates a challenge to analytical tractability. We show how an appropriate formulation of the problem reduces analysis to a tractable extension of linearization methods applied to the spatial analog of the well known costate/state dynamics. We illustrate the usefulness of our methods on bioeconomic applications, but the methods have more general economic applications where spatial considerations are important. We believe that the extension of Turing analysis and the theory associated with dispersion relationship to recursive infinite horizon optimal control settings is new.

JEL Classification Q2, C6

Keywords: Spatial analysis, Pattern formation, Turing mechanism, Optimal control, bioeconomic problems.

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1 Introduction

In economics the importance of space has long been recognized in the context of location theory,[1] although as noted by Paul Krugman (1998) there has been neglect in a systematic analysis of spatial economics, associated mainly with difficulties in developing tractable models of imperfect competition which are essential in the analysis of location patterns. After the early 1990's there was a renewed interest in spatial economics mainly in the context of *new economic geography,*[2] which concentrates on issues such as the determinants of regional growth and regional interactions, or the location and size of cities (e.g. Paul Krugman, 1993).

In environmental and resource management problems the majority of the analysis has been concentrated on taking into account the temporal variation of the phenomena, and has been focused on issues such as the transition dynamics towards a steady state, or the steady-state stability characteristics. However, it is clear that when renewable and especially biological resources are analyzed, the spatial variation of the phenomenon is also important. Biological resources tend to disperse in space under forces promoting "spreading", or "concentrating" (Akira Okubo, 2001); these processes along with intra and inter species interactions induce the formation of spatial patterns for species. In the management of economic-ecological problems, the importance of introducing the spatial dimension can be associated with a few attempts to incorporate spatial issues, such as resource management in patchy environments (James Sanchirico and James Wilen 1999, 2001; Sanchirico, 2004; William Brock and Anastasios Xepapadeas, 2002), the study of control models for interacting species (Suzanne Lenhart and Mahadev Bhat (1992), Lenhart et al. 1999) or the control of

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[1] See for example Alfred Weber (1909), Harold Hotelling (1929), Walter Christaller (1933), and August Löcsh (1940) for early analysis.

[2] Paul Krugman (1998) attributes this new research to: the ability to model monopolistic competition using the well known model of Avinash Dixit and Joseph Stiglitz (1977); the proper modeling of transaction costs; the use of evolutionary game theory; and the use of computers for numerical examples.

surface contamination in water bodies (Bhat et al. 1999)

In the economic-ecological context, a central issue that this paper is trying to explore, is under what conditions interacting processes characterizing movements of biological resources, and economic variables which reflect human effects on the resource (e.g. harvesting effort) could generate steady-state spatial patterns for the resource and the economic variables. That is, a steady-state concentration of the resource and the economic variable which is different at different points in a given spatial domain. We will call this formation of spatial patterns *spatial heterogeneity,* in contrast to *spatial homogeneity* which implies that the steady state concentration of the resource and the economic variable is the same at all points in a given spatial domain.[3]

A central concept in modelling the dispersal of biological resources is that of *diffusion.* Diffusion is defined as a process where the microscopic irregular movement of particles such as cells, bacteria, chemicals, or animals results in some macroscopic regular motion of the group (Okubo and Simon Levin, 2001; James Murray, 1993, 2003). Biological diffusion is based on random walk models, which when coupled with population growth equations, leads to general reaction-diffusion systems.[4] In general a diffusion process in an ecosystem tends to produce a uniform population density, that is spatial homogeneity. Thus it might be expected that diffusion would "stabilize" ecosystems where species disperse and humans intervene through harvesting.

There is however one exception known as *diffusion induced instability*, or *diffusive instability* (Okubo et al., 2001). It was Alan Turing (1952) who suggested that under certain conditions reaction-diffusion systems can generate spatially heterogeneous patterns. This is

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[3] All dynamic models where spatial characteristics and dispersal are ignored leads to spatial homogeneity.

[4] When only one species is examined the coupling of classical diffusion with a logistic growth function leads to the so-called Fisher-Kolmogorov equation.

the so-called *Turing mechanism* for generating diffusion instability.

The purpose of this paper is to explore the impact of the Turing mechanism in the emergence of diffusive instability in unified economic/ecological models of resource management. This is a different approach to the one most commonly used to address spatial issues, which is the use of metapopulation models in discrete patchy environments with dispersal among patches. We believe that the use of the Turing mechanism allows us to analyze in detail conditions under which diffusion could produce spatial heterogeneity and generation of spatial patterns, or spatial homogeneity. Thus the Turing mechanism can be used to uncover conditions which generate spatial heterogeneity in models where ecological variables interact with economic variables. When spatial heterogeneity emerges the concentration of variables of interest (e.g. resource stock and level of harvesting effort), in a steady state, are different in different locations of a given spatial domain. Once the mechanism is uncovered the impact of regulation in promoting or eliminating spatial heterogeneity can also be analyzed.

The importance of the Turing mechanism in spatial economics has been recognized by Masahisa Fujita et al. (1999, chapter 6) in the analysis of core-periphery models. Our analysis extends this approach mainly by: explicit modelling of diffusion processes governing interacting economic and ecological state variables in continuous time space; deriving explicit conditions depending on economic-ecological variables, under which diffusion could generate spatial patterns, and probably more importantly by developing the ideas for the emergence of spatial heterogeneity in an optimizing context by an appropriate modification of Pontryagin's maximum principle.

In this context, first we present a descriptive model where the biomass of a renewable resource (e.g. fish) diffuses in a finite one-dimensional spatial domain, and harvesting effort diffuses in the same domain, attracted in locations where profits per boat are higher. We

examine conditions under which: (i) open-access equilibrium generates traveling waves for the resource biomass, and (ii) the Turing mechanism can induce spatial heterogeneity, in the sense that the steady-state fishing stock and fishing effort are different at different points of the spatial domain. We also show how regulation can promote or eliminate spatial heterogeneity.

Second we consider the emergence of spatial heterogeneity in the context of an optimizing model, where the objective of a social planner is to maximize a welfare criterion subject to resource dynamics that include a diffusion process. We present a suggestion for extending Pontryagin's maximum principle to the optimal control of diffusion. Although conditions for the optimal control of partial differential equations have been derived either in abstract settings (e.g. Jacques-Louis Lions 1971) or for specific problems,[5] our derivation, not only makes the paper self contained, but it is also close to the optimal control formalism used by economists, so it can be used for analyzing other types of economic problems, where state variables are governed by diffusion processes. Furthermore, the Pontryagin principle developed in this paper allows for an extension of the Turing mechanism for generation of spatial patterns, to the optimal control of systems under diffusion.

A new, to our knowledge, characteristic of our continuous space-time approach is that we are able to embed Turing analysis in an optimal control recursive infinite horizon approach in a way that allows us to locate sufficient conditions on parameters of the system (for example, the discount rate on the future, and interaction terms in the dynamics) for diffusive instability to emerge even in systems that are being optimally controlled. This mathematically challenging problem becomes tractible by exploiting the recursive structure of the utility and the dynamics in our continuous space/time framework in contrast to the more traditional approach of discrete patch optimizing models. This is so because the symmetries in the spatial

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[5] See for example Lenhart and Bhat, (1992); Lenhart et al., (1999); Bhat et al., (1999); Jean-Pierre Raymond and Housnaa.Zidani, (1998, 1999)

4

structure coupled with the recursivity in the temporal structure of our framework reduce the potentially very large number of state and costate variables to a pair of "sufficient" variables that describe the dynamics of the whole system. We believe that our framework will be quite easily adaptable to other applications, including an extension of the classical Ramsey Solow growth model to include spatial externalities. Colin Clark's classic volume (1990) as well as the work of Sanchirico and Wilen (1999, 2001) is very suggestive, but they do not contain the unification of Turing analysis with infinite horizon temporally recursive optimal control problems that we present here. We set the stage by analysis of some descriptive frameworks before turning to optimal control counterparts

Here, we use our methodology to study an optimal fishery management problem and a bioinvasion problem under diffusion. For the fishery problem, our results suggest that diffusion could alter the usual saddle point characteristics of the spatially homogeneous steady state as defined by the modified Hamiltonian dynamic system. In an analogue to the Turing mechanism for an optimizing system, spatial heterogeneity in a steady state could be the result of optimal management. On the other hand diffusion could stabilize, in the saddle point sense, an unstable steady state of an optimal control problem. For the bioinvasion problem we develop a most rapid approach path (MRAP) solution to the optimal control of diffusion processes with linear structure, and derive conditions under which it is optimal: to fight the invasion to the maximum when it first occurs; to do nothing at all, or to attain a spatially differentiated target biomass of the invasive species as rapidly as possible.

2 Diffusion and Spatial Heterogeneity in Descriptive Models of Resource Management

2.1 Spatial Open Access Equilibrium with Resource Biomass Diffusion

We start by considering the case where resource biomass diffuses in a spatial domain and harvesting takes place in an open access way. Let $x(z,t)$ denote the concentration of the biomass of a renewable resource (e.g. fish) at spatial point $z \in Z$, at time t. We assume that biomass grows according to a standard growth function $F(x)$ which determines the resource's kinetics but also disperses in space with a constant diffusion coefficient D_x.[6]

$$\frac{\partial x(z,t)}{\partial t} = F(x(z,t)) + D_x \nabla^2 x(z,t) \tag{1}$$

Harvesting $H(z,t)$ of the resource is determined as $H(z,t) = qx(z,t)E(z,t)$, where $E(z,t)$ denotes the concentration of harvesting effort (e.g. boats) at spatial point z and time t, and q is catchability coefficient. Assuming that the harvest is sold at a fixed world price, profits accruing at location z are defined as

$$pqx(z,t)E(z,t) - C(E(z,t)) \tag{2}$$

where $C(E(z,t))$ is the total cost of applying effort $E(z,t)$ at location z. We assume that effort is attracted by profits per boat and that effort (boats) diffuses in the spatial domain infinitely fast so that profits are equated in every site. Then in open access equilibrium with boats allowed to enter from "outside", profits are driven to zero at each site, or

$$pqxE - C(E) = 0 \text{ or } (pqx - AC(E))E = 0 \text{ for all } z \tag{3}$$

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[6] In addition to standard notation we denote derivatives with respect to the spatial variable z, by $\nabla^d y = \frac{\partial^d y}{\partial z^d}$, $d = 1, 2$.

where $AC(E)$ denotes average costs. Assuming linear increasing average cost or $AC(E) = c_0 + (c_1/2)E$, profit dissipation implies, using (3), that effort is determined as

$$\bar{E}(t,z) = \frac{2(pqx(t,z) - c_0)}{c_1} > 0 \text{ if } pqx - c_0 > 0 \tag{4}$$

$$\bar{E}(t,z) = 0 \text{ otherwise} \tag{5}$$

Thus with harvesting, logistic growth $F(x) = x(s - rx)$,[7] and open access equilibrium at all sites, biomass diffuses according to the following Fisher-Kolmogorov equation:[8]

$$\frac{\partial x}{\partial t} = x(s - rx) - qEx + D_x \nabla^2 x \tag{6}$$

or using (4),

$$\frac{\partial x}{\partial t} = s' x(1 - ax) + D_x \frac{\partial^2 x}{\partial z^2} \tag{7}$$

$$s' = \left(s + \frac{2qc_0}{c_1}\right) \ , \ r' = \left(r + \frac{2q^2 p}{c_1}\right) \ , \ a = \frac{r'}{s'} \tag{8}$$

If we introduce harvesting and open access equilibrium at all sites then biomass diffuses according to the Fisher-Kolmogorov equation (7).[9] Following Murray (1993), rescaling (7) by writing $t^* = s' t$ and $z^* = z \left(\frac{s'}{D_x}\right)^{1/2}$ and omitting asterisks, we obtain

$$\frac{\partial x}{\partial t} = x(1 - ax) + \frac{\partial^2 x}{\partial z^2} \tag{9}$$

with spatially homogeneous states 0 and $1/a$, which are unstable and stable respectively. In this case the positive equilibrium carrying capacity is defined as

$$K = \frac{1}{a} \tag{10}$$

As shown by Murray (1993), (9) has a traveling wave solution which can be written as

$$x(z,t) = X(v), \ v = z - ct \tag{11}$$

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[7] We write x instead of $x(z,t)$ to simplify notation.

[8] See Murray (1993 Chapter 11.2 page 277).

[9] See Murray (1993, Chapter 11.2 page 277).

where c is the speed of the wave. For a traveling wave to exist, the speed c must exceed the *minimum* wave speed which under Kolmogorov initial conditions is determined for the dimensional equation (7) by

$$c \geq c_{\min} = 2 \left(s' D_x \right)^{1/2} = 2 \left[\left(s + \frac{2qc_0}{c_1} \right) D_x \right]^{1/2} \tag{12}$$

The wave front solution is depicted in figure 1.

[Figure 1]

These results can be summarized in the following proposition:

Proposition 1 *When biomass disperses in space according to (1), then open access harvesting, with harvesting effort diffusing fast and resulting in zero profit spatial equilibrium, induces convergence to a traveling wave solution for the biomass $X(v)$, with corresponding effort $\bar{E}(v) = \frac{2(pqX(v) - c_0)}{c_1}$.*

From (12) it can be seen that the wave speed depends on both ecological and economic parameters. In particular it is increasing in s, the catchability coefficient q, the initial marginal effort cost c_0, but declining in the slope of marginal effort cost c_1.

Our model can be used to analyze the impact of regulation. Assume that regulation involves linear spatially homogeneous taxes on effort (e.g. number or size of boats) or harvesting. Under an effort tax, zero profit condition and open access effort become

$$pqxE - \tau E - C(E) = 0 \text{ or } [pqx - \tau - AC(E)] E = 0 \text{ for all } z, \tag{13}$$

$$\bar{E}(t, z; \tau) = \frac{2(pqx(t, z) - c_0 - \tau)}{c_1} \tag{14}$$

respectively. Under a linear spatially homogeneous harvesting tax they become

$$pqxE - \tau qEx - C(E) = 0 \text{ or } [(p - \tau)qx - AC(E)] E = 0 \text{ for all } z \tag{15}$$

$$\bar{E}(t, z; \tau) = \frac{2[(p - \tau)qx(t, z) - c_0]}{c_1} \tag{16}$$

8

respectively. Given the above equations the effects of regulation are obtained in the following proposition.

Proposition 2 *A spatially homogeneous linear tax on effort will increase the wave speed c, and the equilibrium carrying capacity K, while a spatially homogeneous linear tax on harvesting will increase the equilibrium carrying capacity K but leave the wave speed c unchanged.*

For Proof see Appendix.

2.2 Biomass-Effort Reaction Diffusion and Pattern Formation

In the previous section we assumed that in an unbounded spatial domain effort diffuses fast to dissipate profits under open access across all sites. In this section we consider a bounded spatial domain $Z = [0, \alpha]$ and we assume that effort does not diffuse infinitely fast in search of profits. This assumption allows us to study the interactions between biomass and effort diffusion and the generation of spatial patterns where biomass and effort exhibit different concentrations.

We assume that effort is attracted by profits per boat and that effort (boats) disperses in the spatial domain with a constant diffusion coefficient D_E. Although boats could move fast in open access property regimes, movements could be restricted in communal property regimes (e.g. Fikret Berkes 1996), where due to institutional arrangements, there is exclusion of boats from certain areas and general frictions in the movement of boats towards the biomass. The structure of the model implies that the movement of biomass and effort in time and space can be described by the following reaction diffusion system

$$\frac{\partial x}{\partial t} = x\left(s - rx\right) - qEx + D_x\nabla^2 x \tag{17}$$

$$\frac{\partial E}{\partial t} = \delta E\left(pqx - AC\left(E\right)\right) + D_E\nabla^2 E \ , \ \delta > 0 \tag{18}$$

$$x(z,0), \ E\left(z,0\right) \text{ given}, \ \nabla x = \nabla E = 0 \text{ for } z = 0, \alpha \tag{19}$$

where $AC\left(E\right)$ is the average cost curve, assumed to be U-shaped. By (19) it is assumed

9

that there is no external biomass or effort input on the boundary of the spatial domain.[10]

Given the system of (17) and (18) we examine conditions under which the Turing mechanism induces diffusive driven instability and creates a heterogeneous spatial pattern of resource biomass and harvesting effort.

2.2.1 Biomass-Effort Spatial Patterns

In analyzing diffusion induced instability we start from a system which, in the absence of diffusion, exhibits stable spatially homogeneous steady states. The spatially homogeneous system of (17) and (18), with $D_x = D_E = 0$ is defined as:

$$\dot{x} = x(s - rx) - qEx \tag{20}$$

$$\dot{E} = \delta E(pqx - AC(E)) \ , \delta > 0 \tag{21}$$

where a steady state $(x^*, E^*) > 0$ for the spatial homogeneous is determined as the solution of $\dot{x} = \dot{E} = 0$. The homogeneous steady state is defined by the intersection of the isocines

$$x|_{\dot{x}=0} = \frac{s}{r} - \frac{q}{r}E \tag{22}$$

$$x|_{\dot{E}=0} = \frac{AC(E)}{pq} \tag{23}$$

where (23) is linear with a negative slope, while (23) is U-shaped with $E^0 = \arg\min AC(E)$ being the effort minimizing average cost. Assume that two steady states E_1^* and E_2^* exist. As shown in figure 3 it holds that

$$0 < E_1^* < E_2^* \text{ where } AC'(E_1^*) < 0, \ AC'(E_2^*) \gtreqless 0. \tag{24}$$

Furthermore, as indicated by the flows of the phase diagram, the high effort steady state is stable while the low effort is unstable.

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[10] This is a zero flux boundary conditions which is imposed so that the organizing pattern between biomass and effort is emerging as a result of their interactions, is self-organizing and not driven by boundary conditions (Murray 2003, Vol II, p.82).

10

[Figure 2]

Linearizing around a steady state (x^*, E^*), [11] the linearized spatial homogeneous system can be written as

$$\dot{\mathbf{w}} = J\mathbf{w} \ , \ \mathbf{w} = \left(\begin{array}{c} x - x^* \\ \\ E - E^* \end{array} \right)$$

where the linearization matrix J around a steady state is defined as

$$J = \left[\begin{array}{cc} -rx^* & -qx^* \\ \\ \delta pqE^* & -\delta E^* AC'(E^*) \end{array} \right] = \left[\begin{array}{cc} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{array} \right] \tag{25}$$

At the stable steady state:

$$tr(J) = \left(-rx^* - \delta E^* AC'(E^*) \right) < 0$$

$$Det(J) = \delta E^* x^* \left(rAC'(E^*) + pq^2 \right) > 0$$

If the stable steady state is at the increasing part of the average cost then $a_{11}a_{22} > 0$, while $a_{12}a_{21} < 0$. If the stable steady state is at the decreasing part of the average cost then $a_{11}a_{22} < 0$, $a_{12}a_{21} < 0$. Since for diffusive instability we require opposite signs between a_{11} and a_{22} and between a_{12} and a_{21}, [12] we consider the high effort steady state occurring at the declining part of $AC(E)$. In this case the sign pattern for J is $(a_{11}, a_{12}) < 0$, $(a_{21}, a_{22}) > 0$. Linearizing the full system (17) and (18) we obtain

$$\mathbf{w}_t = J\mathbf{w} + D\nabla^2\mathbf{w} \ , \tag{26}$$

$$\mathbf{w}_t = \left(\begin{array}{c} \partial x/\partial t \\ \\ \partial E/\partial t \end{array} \right) \ , D = \left(\begin{array}{cc} D_x & 0 \\ \\ 0 & D_E \end{array} \right)$$

Following Murray (2003) we consider the time-independent solution of the spatial eigenvalue problem

$$\nabla^2\mathbf{W} + k^2\mathbf{W} = 0, \ \nabla\mathbf{W} = \mathbf{0}, \text{for } z = 0, a \tag{27}$$

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[11] We follow Murray (2003, Vol II, Ch. 2.3).

[12] Okubo et al., (2001, pp. 350-351).

11

where k is the eigenvalue. For the one-dimensional domain $[0, a]$ we have solutions for (27) which are of the form

$$\mathbf{W}_k(z) = \mathbf{A}_n \cos\left(\frac{n\pi z}{a}\right) \ , \ n = \pm 1, \ \pm 2, ..., \tag{28}$$

where $\mathbf{A}_n$ are arbitrary constants. Solution (28) satisfies the zero flux condition at $z = 0$ and $z = a$. The eigenvalue is $k = n\pi/a$ and $1/k = a/n\pi$ is a measure of the wave like pattern. The eigenvalue k is called the *wavenumber* and $1/k$ is proportional to the wavelength $\omega : \omega = 2\pi/k = 2\alpha/n$. Let $\mathbf{W}_k(z)$ be the eigenfunction corresponding to the wavenumber k, we look for solutions of (26) of the form

$$\mathbf{w}(z, t) = \sum_k c_k e^{\lambda t} \mathbf{W}_k(z) \tag{29}$$

Substituting (29) into (26), using (27) and canceling $e^{\lambda t}$ we obtain for each k or equivalently each n, $\lambda \mathbf{W}_k = J\mathbf{W}_k - Dk^2\mathbf{W}_k$. Since we require non trivial solutions for $\mathbf{W}_k$, λ is determined by

$$\left|\lambda I - J - Dk^2\right| = 0$$

Then the eigenvalue $\lambda(k)$ as a function of the wavenumber is obtained as the roots of

$$\lambda^2 + \left[(D_x + D_E) k^2 - (a_{11} + a_{22})\right] \lambda + h\left(k^2\right) = 0 \tag{30}$$

$$h\left(k^2\right) = D_x D_E k^4 - (D_x a_{22} + D_E a_{11}) k^2 + Det(J) \tag{31}$$

Since the spatially homogeneous steady state (x^*, E^*), is stable it holds that $\operatorname{Re}\lambda\left(k^2 = 0\right) < 0$. For the steady state to be unstable in spatial disturbances it is required that $\operatorname{Re}\lambda(k) > 0$ for some $k \neq 0$. But $\operatorname{Re}\lambda\left(k^2\right) > 0$ only if $h\left(k^2\right) < 0$. The minimum of $h\left(k^2\right)$ occurs at k_c^2 obtained after differentiating (31) as $k_m^2 = \frac{(D_x a_{22} + D_E a_{11})}{2D_x D_E}$ which implies that for diffusive instability we need $h\left(k_m^2\right) < 0$. The final condition for diffusive instability becomes (Okubo et al., 2001)[13]

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[13] The assumption of friction in the boat movements because of institutional reasons, implies that δ is sufficiently low to sustain the spatial pattern.

$$a_{11}D_E + a_{22}D_x > 2\left(a_{11}a_{22} - a_{12}a_{21}\right)^{1/2}\left(D_E D_x\right)^{1/2} > 0 \tag{32}$$

Assuming that this condition is satisfied at the spatially homogeneous steady state, then the spatially heterogeneous solution is the sum of the unstable modes with $\operatorname{Re}\lambda\left(k^2\right) > 0$, or

$$\mathbf{w}\left(z,t\right) \sim \sum_{n_1}^{n_2} \mathbf{C}_n \exp\left[\lambda\left(\frac{n^2\pi^2}{a^2}\right)t\right]\cos\frac{n\pi z}{a}\ ,k^2 = \left(\frac{n\pi}{a}\right)^2 \tag{33}$$

where λ are the positive solutions of the quadratic (30), n_1 is the smallest integer greater or equal to ak_1/π and n_2 is the largest integer less than or equal to ak_2/π. The wavenumbers k_1 and k_2 are such that $k_1^2 < k_m^2 < k_2^2$, with $h\left(k_1^2\right) = h\left(k_2^2\right) = 0$ and $h\left(k^2\right) < 0$ for $k^2 \in \left(k_1^2, k_2^2\right)$. That is, (k_1, k_2) is the range of unstable wavenumbers for which $\operatorname{Re}\lambda\left(k^2\right) > 0$.

To obtain an idea of the solution described by (33), we follow Murray (2003) and assume that the range of unstable wave numbers $\left(k_1^2, k_2^2\right)$ is such there exists only one corresponding $n = 1$, then the only unstable mode is $\cos\left(\pi z/a\right)$ and

$$\mathbf{w}\left(z,t\right) \sim \mathbf{C}_1 \exp\left[\lambda\left(\frac{\pi^2}{a^2}\right)t\right]\cos\frac{\pi z}{a}\ ,k^2 = \left(\frac{\pi}{a}\right)^2 \tag{34}$$

The solution for the biomass and effort assuming small positive $\mathbf{C}_1 = \left(\varepsilon_x, \varepsilon_E\right)'$ take the form

$$x\left(z,t\right) \quad \sim \quad x^* + \varepsilon_x \exp\left[\lambda\left(\frac{\pi^2}{a^2}\right)t\right]\cos\frac{\pi z}{a} \tag{35}$$

$$E\left(z,t\right) \quad \sim \quad E^* + \varepsilon_E \exp\left[\lambda\left(\frac{\pi^2}{a^2}\right)t\right]\cos\frac{\pi z}{a} \tag{36}$$

Since $\lambda\left(\frac{\pi^2}{a^2}\right) > 0$, as t increases the deviation from the spatial homogeneous solution does not die out and could eventually be transformed into a spatial pattern which is like a single cosine mode. If the domain is sufficiently large to include a larger number of unstable wavenumbers then the spatial pattern is more complex. With exponentially growing solutions for all time for (35) and (36), then $x \to \infty$ and $E \to \infty$ as $t \to \infty$ would be implied. However it is assumed that the linear unstable eigenfunctions are bounded by the nonlinear terms and

13

that a spatially heterogeneous steady state will emerge. The main assumption here is the existence of a bounding domain for the kinetics of (17) and (18) in the positive quadrant (Murray, 2003, Vol II p. 87). Thus the bounding set that constrains the kinetics will also contain the solutions (35) and (36) when diffusion is present. Then the growing solution approaches, as $t \rightarrow \infty$, a cosine like spatial pattern, which implies spatial heterogeneity of the steady state. Figure 4, draws on Murray (2003 Vol II, pp. 94-95) to represent one possible spatial pattern for $x(z,t)$. Shaded areas represent spatial biomass concentration above x^*, while non shaded areas represent spatial biomass concentration below x^*.

[Figure 3]

The interactions between effort and biomass are shown in figure 5. Assume that effort increases and reduces biomass below the steady state x^*. This would result in a flux of biomass from neighboring regions which would reduce the effort in these regions, causing fish biomass to increase and so on until a spatial pattern is reached.

[Figure 4]

As we show above the reaction diffusion mechanism characterized by (17) and (18) might be diffusionally unstable, but the solution could evolve to a spatially heterogenous steady state defined by:

$$x_s(z), \ E_s(z) \ \text{as} \ t \rightarrow \infty$$

Then, setting $(\partial x / \partial t, \partial E / \partial t) = 0$ in (17) and (18), we obtain that $x_s(z), \ E_s(z)$ should satisfy

$$0 = x(z)(s - rx(z)) - qE(z)x(z) + D_x x''(z) \tag{37}$$

$$0 = \delta E(z)(pqx(z) - AC(E(z))) + D_E E''(z) \tag{38}$$

$$x'(0) = x'(a) = E'(0) = E'(a) = 0 \tag{39}$$

14

Then a measure of spatial heterogeneity at the steady state is given by the heterogeneity function which is defined as

$$\mathcal{G} = \int_0^a \left(x'^2 + E'^2 \right) dz \geq 0 \tag{40}$$

Integrating by parts (40) and using the zero flux condition (39) we obtain

$$\mathcal{G} = - \int_0^a \left(xx'' + EE'' \right) dz$$

which becomes, using (37) and (38),

$$\mathcal{G} = \int_0^a \left[\frac{x^2}{D_x} \left(s - rx - qE \right) + \frac{\delta E}{D_E} \left(pqx - AC\left(E\right) \right) \right] dz \tag{41}$$

if there is no spatial patterning $s - rx - qE = 0$ and $pqx - AC\left(E\right) = 0$, which are the spatially homogeneous solutions, and $\mathcal{G} = 0$.

2.3 Spatial Heterogeneity and Regulation

As we showed in the previous section, the adaptive biomass-effort system is likely to create spatial heterogeneity under an appropriate institutional regime inducing certain parameter constellation. This implies, for example, that in the case presented in figures 4 and 5 the biomass concentration, effort and profits will be different at different locations of our spatial domain. This can emerge in situations where, because of institutional allocation of the "rights to fish" which restricts boats from certain patches, fish biomass and boat movements are compatible in speed for the Turing mechanism to create spatial patterns and potential spatial inequalities. The measure of inequality can be given for example by the heterogeneity function (40), then social justice would require regulation to support spatial homogeneity. The problem then is reduced to that of finding instruments that will prevent diffusive instability.

As indicated in the previous section, diffusive instability cannot occur if the sign pattern of the linearization matrix (25) does not show opposition of signs between a_{11} and a_{22} and

between a_{12} and a_{21}. Thus given (25), the target is to change the sign structure, through a regulatory instrument, in a way that will prevent diffusive instability. An instrument affecting harvesting behavior will affect profits and consequently the second row of (25).

We consider feedback control instruments in the general form of a non linear tax on effort (e.g. boat size or boat numbers) or on harvesting $\tau(x, E)$, with the property that when the tax is applied, then either a_{21} or a_{22} will change sign so that diffusive instability is not supported.

Proposition 3 *A spatially homogeneous non linear tax on effort of the feedback form $\tau(E)$ with $\tau'(E) > 0$ and $\tau'(E^+) + AC'(E^+) > 0$, where E^+ is the regulated spatially homogeneous steady state for effort, will prevent the emergence of spatial heterogeneity.*

For proof see Appendix.

The effect of the nonlinear tax on effort is to shift the average cost curve, or equivalently the $x|_{\dot{E}=0}$ curve so that the intersection with the $x|_{\dot{x}=0}$ curve, takes place at the increasing part of the average cost curve as shown in Figure 3, where the new curve is AC_{reg}.

A feedback tax on harvesting can also be used as a regulatory instrument.

Proposition 4 *A spatially homogeneous non linear tax on harvesting of the feedback form $\tau(E, x)$ with $p - \tau(E, x) > 0$ $\frac{\partial \tau}{\partial E} > 0, \frac{\partial \tau}{\partial x} < 0$ and $\frac{\partial \tau(E^+, x^+)}{\partial E} qx^+ + AC'(E^+) > 0$, where (E^+, x^+) is the regulated spatially homogeneous steady state for effort, will prevent the emergence of spatial heterogeneity.*

For proof see Appendix.

The effect of the nonlinear tax on effort is to shift the $x|_{\dot{E}=0}$ curve so that the intersection with the $x|_{\dot{x}=0}$ curve takes place at the increasing part of the average cost curve as shown in Figure 3.

It is interesting to note from these two propositions that a feedback tax on harvesting which depends on biomass alone, that is a tax $\tau(x)$, cannot exclude diffusive instability, because in this case the a_{21} element is positive, but the a_{22} element is now $-\delta E^+ AC'(E^+)$. Thus intersections at the decreasing part of the average cost curve cannot be excluded.

16

On the other hand consider the introduction of a new technology, say because of subsidization, that increases the catchability coefficient q, and assume that with the old technology the $x|_{\dot{x}=0}$ isocine was intersecting the $x|_{\dot{E}=0}$ at the increasing part of the average cost curve, point S in figure 2, so that diffusive instability was not possible. The increase in q rotates the $x|_{\dot{x}=0}$ isocine towards the origin so that the new steady state could take place at the decreasing part of the average cost curve. Then, as has been shown above, diffusive instability is possible. Thus,

Proposition 5 *In the model of biomass and effort diffusion described above, an increase in the catchability coefficient might generate spatial hererogeneity.*

3   On the Optimal Control of Diffusion: An Extension of Pontryagin's Principle

In the previous section we analyzed descriptive models of biomass effort diffusion and examined, in the context of these models, the emergence of spatial heterogeneity through the Turing mechanism. In this section we explicitly introduce optimization and we analyze the effects of the optimal control of diffusion processes in the emergence of spatial heterogeneity.

We start by considering the optimal control problem defined in the spatial domain $z \in [z_0, z_1]$

$$\max_{\{u(t,z)\}} \int_{z_0}^{z_1} \int_{t_0}^{t_1} f\left(x\left(t,z\right), u\left(t,z\right)\right) dt dz \tag{42}$$

$$\text{s.t.}\;\; \frac{\partial x\left(t,z\right)}{\partial t} = g\left(x\left(t,z\right), u\left(t,z\right)\right) + D\frac{\partial^2 x\left(t,z\right)}{\partial z^2} \tag{43}$$

$$x\left(t_0, z_0\right) \text{ given}, \;\; \left.\frac{\partial x}{\partial z}\right|_{z_0} = \left.\frac{\partial x}{\partial z}\right|_{z_1} = 0 \;\; \text{zero flux} \tag{44}$$

The first part of (44) provides initial conditions, while the second part is a zero flux condition similar to (19). Problem (42) to (44) has been analyzed in more general forms (e.g. Jacques-Louis Lions, 1971). We however choose to present here an extension of Pontryagin's principle for this problem, because it is in the spirit of optimal control formalism used by

economists, and thus can be used for other applications, but also because it makes the whole analysis in the paper self contained.[14]   Furthermore, as noted in the introduction, the use of Pontryagin's principle in continuous time space allows for a drastic reduction of the dimensionality of the dynamic system describing the phenomenon and makes the problem tractable. Our results are presented below, with proofs in the Appendix.

Maximum Principle under diffusion: Necessary Conditions - Finite time horizon (MPD-FT). *Let $u^* = u^*(t, z)$ be a choice of instrument that solves problem (42) to (44) and let $x^* = x^*(t, z)$ be the associate path for the state variable. Then there exists a function $\lambda(t, z)$ such that for each t and z*

1. $u^* = u^*(t, z)$ *maximizes the generalized Hamiltonian function*

$$\mathcal{H}(x(t, z), u, \lambda(t, z)) = f(x, u) + \lambda \left[g(x, u) + D\frac{\partial x^2}{\partial z^2} \right]$$

or

$$f_u + \lambda g_u = 0 \tag{45}$$

2.

$$\begin{aligned} \frac{\partial \lambda}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial x} - D\frac{\partial^2 \lambda}{\partial z^2} = -\left(f_x + \lambda g_x + D\frac{\partial^2 \lambda}{\partial z^2} \right) \\ \frac{\partial x}{\partial t} &= g(x, u^*) + D\frac{\partial^2 x}{\partial z^2} \end{aligned} \tag{46}$$

evaluated at $u^* = u^*(x(t, z), \lambda(t, z))$.

3.   *The following transversality conditions hold*

$$\lambda(t_1) = 0, \ \frac{\partial \lambda(z_1)}{\partial z} = \frac{\partial \lambda(z_0)}{\partial z} = 0 \tag{47}$$

The result can also be extended to infinite time horizon problems with discounting. In

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[14] Similar conditions have been derived for other cases such as the control of parabolic equations (Jean-Pierr Raymond and Housmaa Zidani.1998,1999), boundary control (Lenhart et al. 1999), or distributed parameter control (Dean Carlson et al. 1991; Lenhart and Bhat 1992)

this case the problem is:

$$\int_{z_0}^{z_1} \int_{t_0}^{\infty} e^{-\rho t} f\left(x\left(t,z\right), u\left(t,z\right)\right) dt dz \quad , \rho > 0. \tag{48}$$

$$s.t \quad \frac{\partial x}{\partial t} = g\left(x\left(t,z\right), u\left(t,z\right)\right) + D\frac{\partial^2 x}{\partial z^2} \tag{49}$$

$$x\left(t_0, z_0\right) \text{ given, } \left.\frac{\partial x}{\partial z}\right|_{z_0} = \left.\frac{\partial x}{\partial z}\right|_{z_1} = 0 \text{ zero flux} \tag{50}$$

Maximum Principle under diffusion: Necessary Conditions - Infinite time horizon (MPD-IT). *Let* $u^* = u^*\left(t,z\right)$ *be a choice of instrument that solves problem (42) to (44) and let* $x^* = x^*\left(t,z\right)$ *be the associate path for the state variable. Then there exists a function* $\lambda\left(t,z\right)$ *such that for each t and z*

1. $u^* = u^*\left(t,z\right)$ *maximizes the generalized current value*

Hamiltonian function $\mathcal{H}\left(x\left(t,z\right), u, \lambda\left(t,z\right)\right) = f\left(x,u\right) + \lambda\left[g\left(x,u\right) + D\frac{\partial x^2}{\partial z^2}\right]$, *or*

$$f_u + \lambda g_u = 0 \tag{51}$$

2.

$$\frac{\partial \lambda}{\partial t} = \rho\lambda - \frac{\partial \mathcal{H}}{\partial x} - D\frac{\partial^2 \lambda}{\partial z^2} = \rho\lambda - \left(f_x + \lambda g_x + D\frac{\partial^2 \lambda}{\partial z^2}\right) \tag{52}$$

$$\frac{\partial x}{\partial t} = g\left(x, u^*\right) + D\frac{\partial^2 x}{\partial z^2} \tag{53}$$

evaluated at $u^* = u^*\left(x\left(t,z\right), \lambda\left(t,z\right)\right)$

3. *The following transversality conditions hold*

$$\frac{\partial \lambda\left(z_1\right)}{\partial z} = \frac{\partial \lambda\left(z_0\right)}{\partial z} = 0 \tag{54}$$

It is clear that the pair of (46) or (52) can characterize the whole dynamic system in continuous time space. Conditions (45) - (47) are essentially necessary conditions. Sufficiency conditions can also be derived by extending sufficiency theorems of optimal control. Proofs are provided in the Appendix.

Maximum Principle under diffusion: Sufficient conditions - Finite time horizon

Assume that functions $f(x, u)$ and $g(x, u)$ are concave differentiable functions for problem (42) to (44) and suppose that functions $x^*(t, z)$, $u^*(t, z)$ and $\lambda(t, z)$ satisfy necessary conditions (45) - (47) and (43) for all $t \in [t_0, t_1]$, $z \in [z_0, z_1]$ and that $x(t, z)$ and $\lambda(t, z)$ are continuous with

$$\lambda(t, z) \geq 0 \text{ for all } t \text{ and } z \tag{55}$$

Then the functions $x^*(t, z)$, $u^*(t, z)$ solve the problem (42) to (44). That is the necessary conditions (45) - (47) are also sufficient.

The result can also be extended along the lines of Arrow's sufficiency theorem. We state here the infinite horizon case.

Maximum Principle under diffusion: Sufficient conditions - Infinite time horizon

Let $\mathcal{H}^0$ denote the maximized Hamiltonian, or

$$\mathcal{H}^0(x, \lambda) = \max_u \mathcal{H}(x, u, \lambda) \tag{56}$$

If the maximized Hamiltonian is a concave function of x for given λ then functions $x^*(t, z)$, $u^*(t, z)$ and $\lambda(t, z)$ that satisfy conditions (45) - (47) and (43) for all $z \in [z_0, z_1]$ and the transversality conditions

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t, z) \geq 0, \quad \lim_{t \to \infty} e^{-\rho t} \lambda(t, z) x(t, z) = 0 \tag{57}$$

solve the problem (42) to (44).

3.1 Optimal Harvesting under Biomass Diffusion

Having established the optimality conditions, we are interested in the implications of diffusion on optimally controlled systems regarding mainly the possibility of emergence of spatial

heterogeneity under optimal control, but also the possibility of diffusion acting as a stabilizing force for unstable steady states under optimal control. Let as before $x(z,t)$ denote the concentration of the biomass of a renewable resource (e.g. fish) at spatial point $z \in Z$, at time t. We assume a one-dimensional domain $0 \leq z \leq a$ with zero flux at $z = 0$ and $z = a$ or $\left.\frac{\partial x}{\partial z}\right|_0 = \left.\frac{\partial x}{\partial z}\right|_a = 0$. Biomass grows according to a standard growth function $F(x)$ and disperses in space with a constant diffusion coefficient D, or

$$\frac{\partial x(z,t)}{\partial t} = F(x(z,t)) - H(z,t) + D\nabla^2 x \tag{58}$$

where harvesting $H(z,t)$ of the resource is determined as $H(z,t) = qx(z,t)E(z,t)$, $E(z,t)$ denotes the concentration of harvesting effort (e.g. boats) at spatial point z and time t, and q is catchability coefficient. The total cost of applying effort $E(z,t)$ at location z is given by an increasing and convex function $c(E(z,t),z)$, so if we apply the effort further from the origin, cost increases. Let benefits from harvesting at each point on space be $S(H(z,t))$.

The optimal harvesting problem is then defined as:

$$\max_{E(z,t)} \int_0^\infty \int_Z e^{-\rho t}\left[S(H(z,t)) - c(E(z,t),z)\right] dt dz \tag{59}$$

$$\text{s.t. } \frac{\partial x(t,z)}{\partial t} = F(x(t,z)) - qx(t,z)E(t,z) + D\frac{\partial x^2(t,z)}{\partial z^2}$$

$$x(0,z) \text{ given, and zero flux on } 0,a$$

Following the section above, MPD-IT implies that the optimal control maximizes the generalized Hamiltonian for each location z,

$$\mathcal{H} = S(H(z,t)) - c(E(z,t),z) + \tag{60}$$

$$\mu(t,z)\left[x(t,z)(s - rx(t,z)) - qx(t,z)E(t,z) + D\frac{\partial x^2}{\partial z^2}\right]$$

Setting $S'(H(z,t)) = p(z) > 0$, necessary conditions for the MPD-IT, omitting t to simplify,

imply

$$\frac{\partial \mathcal{H}}{\partial E\left(z\right)} = p\left(z\right)qx\left(z\right) - c^{'}\left(E\left(z\right)\right) - \mu\left(z\right)qx\left(z\right) \Rightarrow \left(p - \mu\right)qx = c^{'}\left(E\right) \quad (61)$$

$$E^{0}\left(z\right) = E\left(x\left(z\right), \mu\left(z\right)\right),\ E^{0}\left(z\right) \geq 0,\ \text{if } p\left(z\right) - \mu\left(z\right) \geq 0,$$

$$\frac{\partial E}{\partial x} = \frac{\left(p - \mu\right)q}{c^{''}} > 0\ ,\ \frac{\partial E}{\partial \mu} = -\frac{qx}{c^{''}} < 0 \text{ for all } z.$$

Then, the Hamiltonian system becomes:

$$\frac{\partial x}{\partial t} = F\left(x\right) - qxE\left(x, \mu\right) + D\frac{\partial x^2}{\partial z^2} = G_1\left(x, \mu\right) + D\frac{\partial x^2}{\partial z^2} \quad (62)$$

$$\frac{\partial \mu}{\partial t} = \left[\rho - F^{'}\left(x\right) + qE\left(x, \mu\right)\right]\mu - pqE\left(x, \mu\right) - D\frac{\partial \mu^2}{\partial z^2} = G_2\left(x, \mu\right) - D\frac{\partial \mu^2}{\partial z^2} \quad (63)$$

A spatially homogeneous (or "flat") system is defined from (62) and (63) for $D = 0$. A "flat" steady state (x^*, μ^*) for this system is determined as the solution of $\frac{\partial x}{\partial t} = \frac{\partial \mu}{\partial t} = 0$.[15] Given the nonlinear nature of (62) and (63), more then one steady state is expected. Assume that such a steady state with $(x^*, \mu^*) > 0$, $E^0 > 0$ exists, and consider the linearization matrix around the steady state

$$J = \begin{bmatrix} G_{1x}\left(x^*, \mu^*\right) & G_{1\mu}\left(x^*, \mu^*\right) \\ G_{2x}\left(x^*, \mu^*\right) & G_{2\mu}\left(x^*, \mu^*\right) \end{bmatrix} \quad (64)$$

$$G_{1x} = F^{'}\left(x^*\right) - qx^*E\left(x^*, \mu^*\right) - qx^*\frac{\partial E(x^*, \mu^*)}{\partial x}$$

$$G_{1\mu} = -qx^*\frac{\partial E(x^*, \mu^*)}{\partial \mu}$$

$$G_{2x} = \left[-F^{''}\left(x^*\right) + q\frac{\partial E(x^*, \mu^*)}{\partial x}\right]\mu^* - pq\frac{\partial E(x^*, \mu^*)}{\partial x} \quad (65)$$

$$G_{2\mu} = \rho - F^{'}\left(x^*\right) + qx^*E + qx^*\frac{\partial E(x^*, \mu^*)}{\partial x} = \rho - G_{1x}$$

For the flat steady state we have $\mathrm{tr}J = G_{1x} + G_{2\mu} = \rho > 0$. Therefore if $\det J > 0$ the steady state is unstable, while if $\det J < 0$ the steady state has the local saddle point property. In the saddle point case there is a one-dimensional manifold such that for any

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[15] See, for example, Clark (1990) for the analysis of this problem.

initial value of μ there is an initial value for x, such that the system converges to the steady state along the manifold.

To analyze the impact of diffusion we follow section 2.2. We have, for the linearization of the full system (62) and (63):

$$\mathbf{w}_t = J\mathbf{w} + \tilde{D}\nabla^2\mathbf{w} , \qquad (66)$$

$$\mathbf{w} = \begin{pmatrix} x(z,t) - x^* \\ \mu(z,t) - \mu^* \end{pmatrix} , \quad \mathbf{w}_t = \begin{pmatrix} \partial x/\partial t \\ \partial \mu/\partial t \end{pmatrix} , \tilde{D} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

and λ must solve

$$\left| \lambda I - J - \tilde{D}k^2 \right| = 0$$

Then the eigenvalue $\lambda(k)$ as a function of the wavenumber is obtained as the roots of

$$\lambda^2 - \rho\lambda + h\left(k^2\right) = 0 \qquad (67)$$

$$h\left(k^2\right) = -D^2 k^4 - D\left[2G_{1x}\left(x^*, \mu^*\right) - \rho\right]k^2 + \det J \qquad (68)$$

where the roots are given by:

$$\lambda_{1,2}\left(k^2\right) = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4h\left(k^2\right)}\right) \qquad (69)$$

It should be noted that the flat (no diffusion) case corresponds to $k^2 = 0$, so that $h\left(k^2 = 0\right) = \det J$, and $\lambda_{1,2} = \frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4\det J}\right)$. We examine the implication of diffusion in two cases

3.1.1 Case I: The Spatially homogeneous steady state is a saddle point $\lambda_2 < 0 < \lambda_1$ for $k^2 = 0$ - Diffusion generates spatial heterogeneity.

In this case $\det J < 0$ and since furthermore $\text{tr} J > 0$ there is a one-dimensional stable manifold with negative slope. On this manifold and in the neighborhood of the steady state, for any initial value of x there is an initial value of μ such that the spatially homogeneous system converges to the flat steady state (x^*, μ^*). For the optimally-controlled system the

solutions are such that

$$\begin{pmatrix} \hat{x}(z,t) \\ \hat{\mu}(z,t) \end{pmatrix} \sim C_2 \mathbf{v}_2 e^{\lambda_2 t} \text{, for all } z \qquad (70)$$

where C_2 is constant determined by initial conditions and $\mathbf{v}_2$ is the eigenvector corresponding to λ_2.[16] The path for the optimal control E is given by $\hat{E}^0 = E(\hat{x}(z,t), \hat{\mu}(z,t))$ for all z. Under diffusion the smallest root is given by

$$\lambda_2 = \frac{1}{2}\left(\rho - \sqrt{\rho^2 - 4h(k^2)}\right) \qquad (71)$$

1. If $0 < h(k^2) < \rho^2/4$, for some k, then this root becomes real and positive.

2. If $h(k^2) > \rho^2/4$, for some k, then both roots are complex with positive real parts.

In both cases above, the system is unstable with both roots having positive real parts. Therefore if $h(k^2) > 0$ for some k, then the Hamiltonian system is unstable, in the neighborhood of the flat steady state, to spatial perturbations. From (68) the quadratic function $h(k^2)$ is concave, and therefore has a maximum. Furthermore $h(0) = \det J < 0$[17] and $h'(0) = -(2G_{1x} - \rho) > 0$ if the steady state is on the declining part of $F(x)$, or $F'(x^*) < 0$. Then $h(k^2)$ has a maximum for

$$k_{\max}^2 : h'(k_{\max}^2) = 0, \text{ or } k_{\max}^2 = -\frac{[2G_{1x}(x^*,\mu^*) - \rho]}{2D} > 0, \text{ for } (2G_{1x} - \rho) < 0 \qquad (72)$$

If $h(k_{\max}^2) > 0$ or $-D^2 k_{\max}^4 - D[2G_{1x}(x^*,\mu^*) - \rho]k_{\max}^2 + \det J > 0$, there exist two positive roots $k_1^2 < k_2^2$ such that $h(k^2) > 0$ for $k^2 \in (k_1^2, k_2^2)$. Figure 5a depicts $h(k^2)$ for this case. This is the dispersion relationship associated with the optimal control problem.[18]

[Figure 5]

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[16] Since we want the controlled system to converge to the optimal steady state, the constant C_1 associated with the positive root λ_1 is set at zero.

[17] This is because the flat steady state has the saddle point property.

[18] For a detailed analysis of the dispersion relationship in problems without optimization, see Murray, (2003).

When diffusion renders both roots positive, diffusive instability emerges in the optimal control problem, in a way similar to the diffusive instability emerging from the Turing mechanism in systems without optimization. In this case the solution (70) for the controlled system becomes, following section 2.2,

$$\hat{x}(z,t) \quad \sim \quad x^* + \varepsilon_x \sum_{n_1}^{n_2} \mathbf{C}_n \exp\left[\lambda_2\left(\frac{n^2\pi^2}{a^2}\right)t\right]\cos\frac{n\pi z}{a} \ , k^2 = \left(\frac{n\pi}{a}\right)^2 \tag{73}$$

$$\hat{\mu}(z,t) \quad \sim \quad x^* + \varepsilon_\mu \sum_{n_1}^{n_2} \mathbf{C}_n \exp\left[\lambda_2\left(\frac{n^2\pi^2}{a^2}\right)t\right]\cos\frac{n\pi z}{a} \ , \tag{74}$$

where $\lambda_2\left(\frac{n^2\pi^2}{a^2}\right)$ is the root that is positive due to diffusion, n_1 is the smallest integer greater than or equal to ak_1^2/π, and n_2 is the largest integer less than or equal to ak_2^2/π. The path for optimal effort will be $E^0(z,t) = E(\hat{x}(z,t),\hat{\mu}(z,t))$. Since $\lambda_2 > 0$ the spatial patterns do not decay as t increases. Thus, provided that the kinetics of the Hamiltonian system have a confined set, the solution converges at the steady state to a spatial pattern. This implies that for a subset of the spatial domain the resource stock and its shadow value are above the flat steady state levels and for another subset they are below the flat steady state levels, similar to figures 3 or 4. This result can be summarized in the following proposition.

Proposition 6 *For an optimal harvesting system which exhibits a saddle point property in the absence of diffusion, it is optimal, under biomass diffusion and for a certain set parameter values, to have emergence of diffusive instability leading to a spatially heterogeneous steady state for the biomass stock, its shadow value and the corresponding optimal harvesting effort.*

The significance of this proposition, which extends the concept of the Turing mechanism to the optimal control of diffusion, is that spatial heterogeneity and pattern formation, resulting from diffusive instability, might be an optimal outcome under certain circumstances.

For regulation purposes and for the harvesting problem examined above, it is clear that the spatially heterogeneous steady state shadow value of the resource stock, and the corresponding harvesting effort, can be used to define *optimal regional* fees or quotas.

3.1.2 Case II: The Spatially homogeneous steady state is unstable $\text{Re}\,\lambda_{1,2} > 0$ for $k^2 = 0$ - Diffusion Stabilizes

Since $\text{tr}J > 0$, this implies that $\det J > 0$. Let $\Delta_D = \rho^2 - 4\det J > 0$ so the we have two positive real roots at the flat steady state. Diffusion can stabilize the system in the sense of producing a negative root. For the smallest root to turn negative or $\lambda_2 < 0$ it is sufficient that

$$h\left(k^2\right) < 0 \tag{75}$$

The quadratic function (68) is concave, and therefore has a maximum. Furthermore $h\left(0\right) = \det J > 0$ and $h'\left(0\right) = -\left(2G_{1x} - \rho\right) > 0$ if the steady state is on the declining part of $F\left(x\right)$, or $F'\left(x^*\right) < 0$. Thus there is a root $k_2^2 > 0$ (see figure 5b) such that for $k^2 > k_2^2$, we have $\lambda_2 < 0$. The solutions for $x\left(z,t\right)$ and $\mu\left(z,t\right)$, will be determined by the sum of exponentials of λ_1 and λ_2. Since we want to stabilize the system we set the constant associated with the positive root λ_1 equal to zero. Then the solution will depend on the sum of unstable and stable modes associated with λ_2. Following previous results the solutions for x and μ will be of the form:

$$\begin{pmatrix} \hat{x}\left(z,t\right) \\ \hat{\mu}\left(z,t\right) \end{pmatrix} \sim \begin{pmatrix} x^* \\ \mu^* \end{pmatrix} + \sum_0^{n_2} \mathbf{C}_{\hat{n}} \exp\left[\lambda\left(\frac{n^2\pi^2}{a^2}\right)t\right] \cos\frac{n\pi z}{a} + \tag{76}$$
$$\sum_{n_2}^{N} \mathbf{C}_n \exp\left[\lambda\left(\frac{n^2\pi^2}{a^2}\right)t\right] \cos\frac{n\pi z}{a},$$

where n_2 is the smallest integer greater or equal to ak_2^2/π and $N > n_2$ and we choose optimal effort such that $E^0\left(z,t\right) = E\left(\hat{x}\left(z,t\right), \hat{\mu}\left(z,t\right)\right)$. Since $\lambda\left(\frac{n^2\pi^2}{a^2}\right) < 0$ for $n > n_2$, all the modes of the third term of (76) decay exponentially. So to converge to the steady state we need to set $\mathbf{C}_{\hat{n}} = 0$, then the spatial patterns corresponding to the third term of (76) will die out with the passage of time and the system will converge to the spatially homogeneous steady state.

This result can be summarized in the following proposition.

Proposition 7 *For an unstable steady state, in the absence of diffusion, of an optimal harvesting problem, it is optimal, under biomass diffusion and for a certain set of parameter values, to stabilize the steady state. Stabilization is in the form of saddle point stability where spatial patterns decay and the system converges along one direction to the previously unstable spatially homogeneous steady state.*

The significance of this proposition is that it shows that under diffusion it is optimal to stabilize a steady state which was unstable under spatial homogeneity.

3.2   On the Optimal Control of Bioinvasions

The framework of the optimization methodology developed in section 3 can be applied to the study of bioinvasion problems which typically involve, along with the temporal dynamics, diffusion in space of an invasive species (e.g. insects). Let the evolution of the biomass of the invasive species given by the diffusion equation

$$\frac{\partial x\left(z,t\right)}{\partial t} = F(x\left(z,t\right),a) - h\left(z,t\right) + D\nabla^2 x \tag{77}$$

where $x\left(0,0\right) = x_0$ denotes the "propagule" of the invasive species which is released at time $t = 0$ at the origin of a one-dimensional space Z. The biological growth function of the invasive species is given by $F(x\left(z,t\right),a)$, with a reflecting general environmental interaction with the species in question, and $h\left(z,t\right)$ denoting the removal (harvesting) of the invasive species, through for example spraying.

Let $c_1\left(z\right)h\left(z,t\right)$ be the cost of removing $h\left(z,t\right)$ from the invasive species at time t and location z, thus $c_1\left(z\right)$ is the site dependent unit removal cost, and $c_2\left(x\left(z,t\right),z\right)$ the cost or damage associated with the amount of biomass $x\left(z,t\right)$ from the invasive species at time t and site z. This cost could be, for example, losses in agricultural production, or treatment cost of those affected by the invasive species.

The bioinvasion control problem for a regulator would be to choose a removal policy $\{h\left(z,t\right)\}$ in space/time to minimize the present value of removal and harvesting costs. The

27

problem can be defined as

$$\min_{\{h(z,t)\}} \int_Z \int_0^\infty e^{-\rho t} \left[c_1(z) h(z,t) + c_2(x(z,t),z) \right] dt dz \qquad (78)$$

$$\text{s.t. (77) and } x(0,0) = x_0 \qquad (79)$$

We exploit the linearity of the objective function and the species dynamics in h to develop

a MRAP for the optimal control of diffusions with a linear structure in the control. The

MRAP essentially determines an optimal or target invasive species biomass, $x^*(z) \geq 0$ in

the following way.

Proposition 8 *The optimal target biomass $x^*(z) \geq 0$ of the invasive species for any $t \in (0, \infty)$ and site $z \in Z = [0, Z_B]$, when the flux of the invasive species on the boundary Z is such that $\frac{\partial x(t,0)}{\partial z} = \frac{\partial x(t,Z_B)}{\partial z}$,[19] is determined as*

$$x^*(z) = \arg\min \{ c_1(z) [F(x(z),a) - \rho x(z)] + c_2(x(z),z) \} \qquad (80)$$

For proof see Appendix.

Objective (80) is now written in MRAP form which is in the form of a sum of independent

terms across space and time which suggests that the optimal thing to do is to approach as

rapidly as possible, for each site z, the desired "target" $x^*(z)$ described by optimization (80).

If we assume that the objective function in (80) is convex in x for all z in Z, then we can

derive some concrete results.

First, it will be optimal to fight an initial bioinvasion, described by $x_0(z)$, as much as

possible, if $x^*(z) = 0$ for all z, when (81) below holds,

$$c_1(z)(F'(0,a) - \rho) + c_2'(0,z) \geq 0, \ \forall z \in Z \qquad (81)$$

With the objective (80) convex in x for each z, condition (81) is sufficient for "fighting to

the max" to be optimal. Condition (81) is easy to interpret. Assume to simplify things that

---

[19] *This assumption covers the zero flux condition used earlier in this paper.*

$F(x, z) = xs(z)$, then condition (81) is

$$c_2'(0, z)/[\rho - s(z)] \geq c_1(z), \ \forall z \in Z \tag{82}$$

which says that you "fight to the max" the bioinvasion, when the capitalized sum of marginal damages from the biomass of the invasive species (adjusted for the growth rate of the species) is greater than the cost of killing one.

Secondly, it is optimal to do nothing if

$$c_2'(k(z), z)/[\rho - s(z)] \leq c_1(z), \ \forall z \in Z \tag{83}$$

where $k(z)$ is the carrying capacity of the invasive species at site z. Equation (83) makes sense from the economic point of view. If $c_2(x)$ is convex, marginal damage is increasing in x. Hence, if the largest possible marginal damage is still not enough when capitalized (at the rate adjusted for growth) to cover the current marginal cost of removing one unit of x, it makes sense that it would be optimal to do nothing in the face of bioinvasion.

Third, it is easy to locate the necessary and sufficient conditions for an interior solution to be optimal, given the assumed convexity of the objective function in (80). The optimality condition for the interior solution, $x^*(z)$, will be

$$c_1(z)(F'(x^*(z), a) - \rho) + c_2'(x^*(z), z) \geq 0, \ \forall z \in Z \tag{84}$$

Then the MRAP to the bioinvasion control, across sites $z \in Z$ and for every t, will take the form

$$h^*(z) = \begin{cases} 0 & \text{if} \quad x(z) < x^*(z) \\ h^{\max} & \text{if} \quad x(z) > x^*(z) \\ F(x^*(z), a) - D\nabla^2 x^*(z) & \text{if} \quad x(z) = x^*(z) \end{cases} \tag{85}$$

Conditions (85) represent an extension of the standard MRAP solution to continuous time space.

4 Conclusions and Suggestions for Future Research

This paper develops methods of control of spatial dynamical systems. In particular it adapts Turing analysis for diffusive instability to bioeconomic problems, and it furthermore extends this analysis to recursive optimal control frameworks. It applies the methods to economic problems of optimal harvesting and optimal control of bioinvasions as illustrations of the potential power of the methods.

In section two of the paper, we formulate a spatial harvesting model in continuous space-time of the Clark (1990), Sanchirico and Wilen (1999, 2001) type, in order to illustrate how the interaction of the Turing mechanism with economic forces can produce travelling wave solutions and spatial heterogeneity in an analytically tractible descriptive framework. We show how this framework could be used to study the interaction of various tax and regulatory policies with the economic dynamics and the biomass dynamics over space-time to produce or to moderate emergent spatial heterogeneity. We use this framework to expose the key role of the dispersion relation in the study of emergent spatial heterogeneity. In section three of the paper, we develop a version of the Pontryagin Maximum Principle for continuous space-time problems that should be useful to economists. While this is not entirely new, we believe the tractible and transparent way in which we develop it should be useful to our fellow economists and other researchers.

We illustrate its usefulness by applying the method to an optimal version of the descriptive problem in section two. We develop the analog of the dispersion relationship for a recursive infinite horizon version of the optimal harvesting problem, and show how one can now easily do local stability analysis by linearization of the analog of the familiar Hamiltonian. We extend the theory associated with the dispersion relationship in descriptive spatial Turing analysis to our recursive infinite horizon optimization framework. This is an extension to

continuous space-time of the economist's familiar infinite horizon framework. We believe this extension of the dispersion relation and Turing analysis is new. In any event it should be useful for many applications involving optimization over space-time in economics and related subjects.

We locate sufficient conditions for diffusion and optimization to induce local instability and, hence, spatial heterogeneity in an originally spatially-homogeneous situation. We also locate sufficient conditions for diffusion and optimization to stabilize and to homogenize an originally spatially-heterogeneous situation.

In addition we show how the optimization framework can be easily modified to analyze "bang-bang" problems which are linear in the control, e.g. MRAP. We illustrate the potential usefulness of this modification to the optimal control of bioinvasions. We give simple, economically-interpretable sufficient conditions for the optimality of various bioinvasion fighting strategies such as "fighting to the max."

We believe that the analytical methods developed in this paper not only provide some useful insights on the optimal control in time-space of some important bioeconomic problems, such as fishery management and control of bioinvasions, but that they can also provide a solid basis for the analysis of a variety of economics problems where spatial considerations are important.

Appendix

Proof of Proposition 2: From the definition of s' and r' we have the following conditions under regulation

$$\text{Effort tax:} \qquad s' = \left(s + \frac{2q(c_0 + \tau)}{c_1}\right) \;,\; r' = \left(r + \frac{2q^2 p}{c_1}\right) \;,\; K = \frac{1}{a} = \frac{s'}{r'}$$

$$\text{Harvesting tax:} \qquad s' = \left(s + \frac{2qc_0}{c_1}\right) \;,\; r' = \left(r + \frac{2q^2(p - \tau)}{c_1}\right) \;,\; K = \frac{1}{a} = \frac{s'}{r'}$$

Thus under the effort tax s', $c_{\min}$ in (12) and K increase, while under the harvesting tax only r' declines and thus K increases. ■

Proof of Proposition 3:Under the tax the evolution of effort for the spatially homogeneous system is described by

$$\dot{E} = \delta E \left(pqx - \tau(E) - AC(E)\right)$$

The regulated homogeneous steady state is defined by the intersection of the isocines

$$x|_{\dot{x}=0} = \frac{s}{r} - \frac{q}{r}E \tag{86}$$

$$x|_{\dot{E}=0} = \frac{\tau(E) + AC(E)}{pq} \tag{87}$$

Then the linearization matrix at the spatially homogeneous steady state becomes

$$J^+ = \begin{bmatrix} -rx^+ & -qx^+ \\ \delta pqE^+ & -\delta E^* \left[\tau'(E^+) + AC'(E^+)\right] \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : \begin{bmatrix} - & - \\ + & - \end{bmatrix} \tag{88}$$

It is clear that $tr(J^+) < 0$ and $Det(J^+) > 0$ so the regulated steady state is stable and because of the sign order at the steady state no diffusive instability is possible.■

Proof of Proposition 4:Under the tax the evolution of effort for the spatially homogeneous system is described by

$$\dot{E} = \delta E \left(pqx - \tau(x, E)qx - AC(E)\right) \tag{89}$$

The regulated homogeneous steady state is defined by the intersection of the isocines

$$x|_{\dot{x}=0} \;=\; \frac{s}{r} - \frac{q}{r}E \tag{90}$$

$$x|_{\dot{E}=0} \;=\; \frac{\tau(E) + AC(E)}{(p - \tau(x, E))\, q} \tag{91}$$

Then the linearization matrix at a spatially homogeneous steady state becomes

$$J^{+} = \begin{bmatrix} -rx^{+} & -qx^{+} \\ \delta\left(p - \tau - \frac{\partial \tau}{\partial x}x^{+}\right)qE^{+} & -\delta E^{*}\left[\frac{\partial \tau}{\partial E}qx^{+} + AC'(E^{+})\right] \end{bmatrix}, \text{ with } \begin{bmatrix} - & - \\ + & - \end{bmatrix}$$

It is clear that $tr(J^{+}) < 0$ and $Det(J^{+}) > 0$ so the regulated steady state is stable and because of the sign order at the steady state no diffusive instability is possible. ∎

Extension of Pontryagin's Principle: Necessary conditions

We develop a variational argument along the lines of Morton Kamien and Nancy Schwartz (1981, pp. 115-116). Problem (42) to (44) can be written as:

$$\begin{aligned} J \;=\; & \int_{z_0}^{z_1}\int_{t_0}^{t_1} f\left(x(t,z), u(t,z)\right) dt\, dz = \int_{z_0}^{z_1}\int_{t_0}^{t_1}\left\{f\left(x(t,z), u(t,z)\right)\right.\\ & \left. \lambda(t,z)\left[g\left(x(t,z), u(t,z)\right) + D\frac{\partial^2 x}{\partial z^2} - \frac{\partial x}{\partial t}\right]\right\} dt\, dz \end{aligned} \tag{92}$$

We integrate by parts the last two terms of (92). The $\lambda(t,z)\frac{\partial x}{\partial t}$ term becomes

$$\begin{aligned} (-1)\int_{z_0}^{z_1}\int_{t_0}^{t_1}\lambda(t,z)\frac{\partial x}{\partial t}dt \;=\; \\ \int_{z_0}^{z_1}\left[-\lambda(t_1)x(t_1) + \lambda(t_0)x(t_0) + \int_{t_0}^{t_1}x(t,z)\frac{\partial \lambda}{\partial t}dt\right]dz \end{aligned} \tag{93}$$

The $\lambda(t,z)D\frac{\partial^2 x}{\partial z^2}$ becomes

$$\begin{aligned} D\int_{z_0}^{z_1}\int_{t_0}^{t_1}\lambda(t,z)\frac{\partial^2 x}{\partial z^2} \;=\; \\ D\int_{t_0}^{t_1}\left[\lambda(z_1)\frac{\partial x}{\partial z}\bigg|_{z_1} - \lambda(z_0)\frac{\partial x}{\partial z}\bigg|_{z_0} - \int_{z_0}^{z_1}\frac{\partial x}{\partial z}\frac{\partial \lambda}{\partial t}dz\right]dt \;=\; \\ -D\int_{t_0}^{t_1}\left[\int_{z_0}^{z_1}\frac{\partial x}{\partial z}\frac{\partial \lambda}{\partial t}dz\right]dt \quad \text{by zero flux} \end{aligned} \tag{94}$$

33

integrating by parts once more we have

$$
(-1) D \int_{t_0}^{t_1} \left[\int_{z_0}^{z_1} \frac{\partial x}{\partial z} \frac{\partial \lambda}{\partial t} dz \right] dt =
$$

$$
D \int_{t_0}^{t_1} \left[-\frac{\partial \lambda(z_1)}{\partial z} x(z_1) + \frac{\partial \lambda(z_0)}{\partial z} x(z_0) \right.
$$

$$
\left. + \int_{z_0}^{z_1} x(t,z) \frac{\partial^2 \lambda}{\partial z^2} \right] dz dt \tag{95}
$$

Thus (92) becomes

$$
\int_{z_0}^{z_1} \int_{t_0}^{t_1} f(x(t,z), u(t,z)) \, dt dz =
$$

$$
\int_{z_0}^{z_1} \int_{t_0}^{t_1} [f(x(t,z), u(t,z)) + \lambda(t,z) [g(x(t,z), u(t,z))]
$$

$$
+ x(t,z) \frac{\partial \lambda}{\partial t} + x(t,z) D \frac{\partial^2 \lambda}{\partial z^2} \right] dt dz
$$

$$
+ \int_{z_0}^{z_1} [-\lambda(t_1) x(t_1) + \lambda(t_0) x(t_0)] dz +
$$

$$
D \int_{t_0}^{t_1} \left[-\frac{\partial \lambda(z_1)}{\partial z} x(z_1) + \frac{\partial \lambda(z_0)}{\partial z} x(z_0) \right] dt \tag{96}
$$

We consider a one parameter family of comparison controls $u^*(t,z) + \varepsilon \eta(t,z)$, where $u^*(t,z)$ is the optimal control, $\eta(t,z)$ is a fixed function and ε is a small parameter. Let $y(t,z,\varepsilon)$, $t \in [t_0, t_1]$, $z[z_0, z_1]$ be the state variable generated by (43) and (44) with control $u^*(t,z) + \varepsilon \eta(t,z)$, $t \in [t_0, t_1]$, $z \in [z_0, z_1]$. We assume that $y(t,z,\varepsilon)$ is a smooth function of all its arguments and the ε enters parametrically. For $\varepsilon = 0$ we have the optimal path $x^*(t,z)$, furthermore all comparison paths must satisfy initial and zero flux conditions. Thus,

$$
y(t,z,0) = x^*(t,z), y(t_0, z_0, \varepsilon) = x(t_0, z_0), \left. \frac{\partial y}{\partial z} \right|_{z_0} = \left. \frac{\partial y}{\partial z} \right|_{z_1} = 0 \tag{97}
$$

When the functions u^*, x^* and η are held fixed the value of (42) evaluated along the control function $u^*(t,z) + \varepsilon \eta(t,z)$ and the corresponding state function $y(t,z,\varepsilon)$ depends only on the single parameter ε. Therefore,

$$
J(\varepsilon) = \int_{z_0}^{z_1} \int_{t_0}^{t_1} [f(y(t,z,\varepsilon), u^*(t,z) + \varepsilon \eta(t,z))] \, dt dz \tag{98}
$$

34

or using (96)

$$
\begin{aligned}
J\left(\varepsilon\right) \;\; = \;\; & \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left[f\left(y\left(t,z,\varepsilon\right), u^*\left(t,z\right) + \varepsilon\eta\left(t,z\right)\right) \right. \\
& + \lambda\left(t,z\right) g\left(y\left(t,z,\varepsilon\right), u^*\left(t,z\right) + \varepsilon\eta\left(t,z\right)\right) \\
& \left. + y\left(t,z,\varepsilon\right) \frac{\partial\lambda}{\partial t} + Dy\left(t,z,\varepsilon\right) \frac{\partial^2\lambda}{\partial z^2} \right] dt dz \\
& + \int_{z_0}^{z_1} \left[-\lambda\left(t_1\right) y\left(t_1,\varepsilon\right) + \lambda\left(t_0\right) y\left(t_0,\varepsilon\right) \right] dz \\
& + D \int_{t_0}^{t_1} \left[-\frac{\partial\lambda\left(z_1\right)}{\partial z} y\left(z_1,\varepsilon\right) + \frac{\partial\lambda\left(z_0\right)}{\partial z} y\left(z_0,\varepsilon\right) \right] dt \qquad (99)
\end{aligned}
$$

Since u^* is a maximizing control the function $J\left(\varepsilon\right)$ assumes the maximum when $\varepsilon = 0$. Thus

$\left. \frac{dJ(\varepsilon)}{d\varepsilon} \right|_{\varepsilon=0}$ or

$$
\begin{aligned}
\left. \frac{dJ\left(\varepsilon\right)}{d\varepsilon} \right|_{\varepsilon=0} = & \\
\int_{z_0}^{z_1} \int_{t_0}^{t_1} & \left[\left(f_x + \lambda g_x + \frac{\partial\lambda}{\partial t} + D\frac{\partial^2\lambda}{\partial z^2} \right) y_\varepsilon + \left(f_u + \lambda g_u \right) \eta \right] dt dz + \\
\int_{z_0}^{z_1} & \left[-\lambda\left(t_1\right) y_\varepsilon\left(t_1,\varepsilon\right) + \lambda\left(t_0\right) y_\varepsilon\left(t_0,\varepsilon\right) \right] dz + \\
D \int_{t_0}^{t_1} & \left[-\frac{\partial\lambda\left(z_1\right)}{\partial z} y_\varepsilon\left(z_1,\varepsilon\right) + \frac{\partial\lambda\left(z_0\right)}{\partial z} y_\varepsilon\left(z_0,\varepsilon\right) \right] dt = 0 \qquad (100)
\end{aligned}
$$

Using the usual transversality condition on λ, we have that $\lambda\left(t_1\right) = 0$, furthermore $y_\varepsilon\left(t_0,\varepsilon\right) = 0$, since $y\left(t_0,\varepsilon\right) = x\left(t_0,z_0\right)$ fixed. If we impose zero flux conditions on λ, then, $\frac{\partial\lambda(z_1)}{\partial z} = \frac{\partial\lambda(z_0)}{\partial z} = 0$, then we obtain from (100)

$$
\frac{\partial\lambda}{\partial t} \;\; = \;\; -\left(f_x + \lambda g_x + D\frac{\partial^2\lambda}{\partial z^2} \right) \qquad (101)
$$

$$
f_u + \lambda g_u = 0 \qquad (102)
$$

So if we define a generalized Hamiltonian function

$$
\mathcal{H} = f\left(x,u\right) + \lambda \left[g\left(x,u\right) + D\frac{\partial x^2}{\partial z^2} \right] \qquad (103)
$$

then by (102) optimality conditions become conditions (45) - (47). The infinite horizon case with discounting is obtained by following Kenneth Arrow and Mordecai Kurz (1970, Chapter II.6). ■

Extension of Pontryagin's Principle: Sufficiency

Suppose that x^*, u^*, λ satisfy conditions (45) - (47) and (43) and let x, u functions satisfying (43). Let f^*, g^* denote functions evaluated along (x^*, u^*) and let f, g denote functions evaluated along the feasible path (x, u). To prove sufficiency we need to show that

$$W \equiv \int_{z_0}^{z_1} \int_{t_0}^{t_1} (f^* - f)\, dtdz \geq 0 \tag{104}$$

From the concavity of f it follows that

$$f^* - f \geq (x^* - x) f_x^* + (u^* - u) f_u^* \tag{105}$$

then

$$W \geq \int_{z_0}^{z_1} \int_{t_0}^{t_1} [(x^* - x) f_x^* + (u^* - u) f_u^*]\, dtdz \tag{106}$$

$$= \int_{z_0}^{z_1} \int_{t_0}^{t_1} \left[(x^* - x) \left(-\frac{\partial \lambda}{\partial t} - \lambda g_x^* - D \frac{\partial^2 \lambda}{\partial z^2} \right) + (u^* - u)(-\lambda g_u^*) \right] dtdz \tag{107}$$

$$= \int_{z_0}^{z_1} \int_{t_0}^{t_1} \lambda \left[g^* - g - (x^* - x) g_x^* - (u^* - u) \right] g_u^* dtdz \geq 0 \tag{108}$$

Condition (107) follows from (106) by using conditions (45) and (46) to substitute for f_x^* and f_u^*. Condition (108) is derived by integrating first by parts the terms involving $\frac{\partial \lambda}{\partial t}$, substituting for $\frac{\partial x}{\partial t}$ from (43), and using the transversality conditions, as has been done in (93), then by integrating twice the terms involving $\frac{\partial^2 \lambda}{\partial z^2}$ and using the zero flux condition as has been done in (95). The non-negativity of the integral in (108) follows from (55) and the concavity of g.

The result can be easily extended along the lines of Arrow's sufficiency theorem (Arrow and Kurz, 1970, Chapter II.6). ∎

Proof of Proposition 8: Substitute h from (77) to (78) and then integrate by parts to eliminate the dx/dt term. For this term we obtain

$$\int_0^\infty \{dx/dt \exp(-\rho t)\} dt = \lim_{T \to \infty} x(T) \exp(-\rho T) - xo + \int_0^\infty \{\rho x \exp(-\rho t)\} dt \tag{109}$$

36

where the "admissible" class of $x(t, z)$ is such that

$$\lim_{T \to \infty} x(T) \exp(-\rho T) = 0 \ \forall z \in Z \tag{110}$$

Integrating the term, $D\frac{\partial^2 x}{\partial z^2}$, over $z \in Z$, we obtain

$$\int_Z \left[D\frac{\partial^2 x}{\partial z^2} \right] dz = D \left. \frac{\partial x}{\partial z} \right|_{bdry(Z)} = D \left(\frac{\partial x(t, B)}{\partial z} - \frac{\partial x(t, 0)}{\partial z} \right) \tag{111}$$

where "$bdry(Z)$" denotes the "boundary of Z". If Z is, for example, a circle $[0, 2\pi]$ where the derivatives are equal at the "boundary points" 0 and 2π, then the equal flux condition $\frac{\partial x(t,0)}{\partial z} = \frac{\partial x(t,B)}{\partial z}$, implies $\frac{\partial x(t,0)}{\partial z} = \frac{\partial x(t,2\pi)}{\partial z}$ and (111) is zero. Thus, for this special case, the diffusion term vanishes in the objective (78). This will be true for any space Z where we assume that the values of the derivatives $\frac{\partial x}{\partial z}$ are equal at the boundaries. If Z is the infinite real line, let $-B, +B$ be large in absolute value, the equal flux assumption implies $\lim_{B \to \infty} \left[\frac{\partial x(t,B)}{\partial z} - \frac{\partial x(t,-B)}{\partial z} \right] = 0$.

Thus, collecting all this together we may write the objective as

$$TC(0) = \int \int \{\exp(-\rho t)\{c_1(z)[F(x, z) - \rho x] + c_2(x, z)\} dz dt + \int \{c_1(z)x_0(z)\}.dz \tag{112}$$

$$J(x, z) \quad : \quad = \{c_1(z)[F(x, a) - \rho x] + c_2(x, z)\} \tag{113}$$

Once we get (113) we can optimize "term by term" to obtain (80). $\blacksquare$

References

Arrow, Kenneth. and Kurz, Mordecai. *Public Investment, the Rate of Return, and Optimal Fiscal Policy*, Johns Hopkins University Press for Resources for the Future: Baltimore, 1970.

Berkes, Fikret. "Social systems, ecological systems, and property rights." in Susan Hanna, Carl, Folke and Karl-Goran, Mäler, eds., *Rights to Nature*, Washington, DC: Island Press, pp. 87-107, 1996.

Brock, William and Xepapadeas, Anastasios. "Optimal Ecosystem Management when Species Compete for Limiting Resources." *Journal of Environmental Economics and Management*, September 2002, 44(2), pp. 189-230.

Carlson, Dean; Haurie, Alain and Leizarowitz, Arie. *Infinite Horizon Optimal Control*, Second Edition, Springer-Verlag: Berlin, 1991.

Clark, Colin. *Mathematical Bioeconomics: The Optimal Management Of Renewable Resources,* Second Edition, New York: Wiley, 1990.

Christaller, Walter. *Central Places in Southern Germany.* Fischer: Jena, 1933.

Dixit, Avinash and Stiglitz, Joseph. "Monopolistic Competition and Optimum Product Diversity." *American Economic Review*, 1977, 67(3), pp. 297-308.

Fujita, Masahisa; Krugman Paul and Venables, Anthony. *The Spatial Economy*, Cambridge Massachusetts: The MIT Press, 1999.

Hotelling, Harold. "Stability in Competition," *Economic Journal*, 1929, 39, pp. 41-57.

Krugman, Paul. "On the Number and Location of Cities." *European Economic Review,* April 1993(2-3), 37, pp. 293-298.

Krugman, Paul. "Space: The Final Frontier," *Journal of Economic Perspectives*, Spring 1998, 12(2), pp. 161-174.

Lions, Jacques- Louis. *Optimal Control of Systems Governed by Partial Differential Equations*, New York: Springer - Verlag, 1971.

Lösch, August. *The Economics of Location.* Fisher: Jena, 1940

Murray, James. *Mathematical Biology*, Second Edition, Berlin, Springer, 1993

Murray, James. *Mathematical Biology*, Third Edition, Berlin, Springer, 2003

Okubo, Akira. "Introduction: The Mathematics of Ecological Diffusion." in Akira Okubo and Simon Levin, eds., *Diffusion and Ecological Problems: Modern Perspectives*, 2nd Edition, Berlin, Springer, 2001.

Okubo, Akira; Alan, Hastings and Powell, Thomas. "Population Dynamics in Temporal and Spatial Domains." in Akira Okubo and Simon Levin, eds., *Diffusion and Ecological Problems: Modern Perspectives*, 2nd Edition, Berlin, Springer, 2001.

Okubo, Akira and Levin, Simon. "The Basics of Diffusion." in Akira Okubo and Simon Levin, eds., *Diffusion and Ecological Problems: Modern Perspectives*, 2nd Edition, Berlin, Springer, 2001.

Kamien, Morton. and Schwartz, Nancy. *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, Second Edition, New York, North-Holland, 1991.

Raymond, Jean-Pierre and Zidani, Housnaa. "Pontryagin's Principle for State-Constrained Control Problems Governed by Parabolic Equations with Unbounded Controls." *SIAM Journal of Control and Optimization,* November 1998 Vol. 36(6), pp. 1853-1879.

—————————. "Hamiltonian Pontryagin's Principles for Control Problems Governed by Semilinear Parabolic Equations." *Applied Mathematics Optimization,* 1999 39 pp. 143–177.

Sanchirico, James. "Additivity Properties of Metapopulation Models: Implications for the Assessment of Marine Reserves, *Resources for the Future*, Discussion Paper 02-66, *Journal of Environmental Economics and Management,* In Press, Available online 21 August 2004,.

Sanchirico, James and Wilen, James. "Bioeconomics of Spatial Exploitation in a Patchy Environment." *Journal of Environmental Economics and Management*, Martch 1999, 37(2), pp. 129-150.

——————. "A Bioeconomic Model of Marine Reserve Creation." *Journal of Environmental Economics and Management*, November 2001, 42(3), pp. 257-276.

Turing, Alan. "The Chemical Basis of Morphogenesis." *Philosophical Transactions of the Royal Society of London*, 1955, 237, pp37-72.

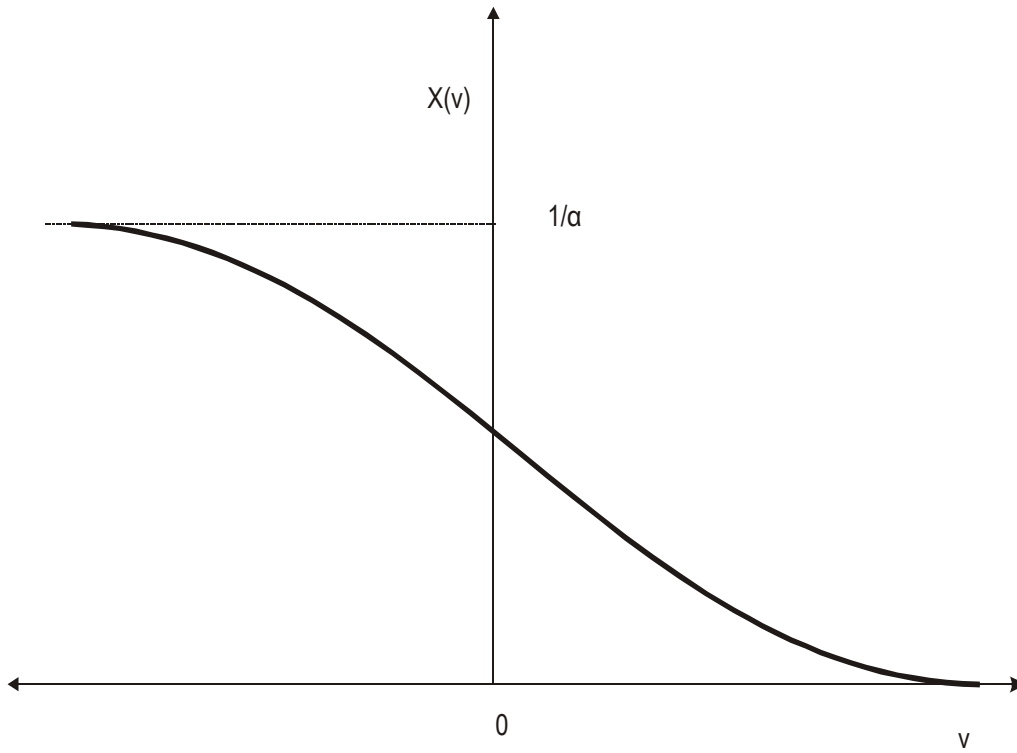**Weber, Alfred.** *Theory of the Location of Industries.* Chicago: University of Chcago Press, 1909.

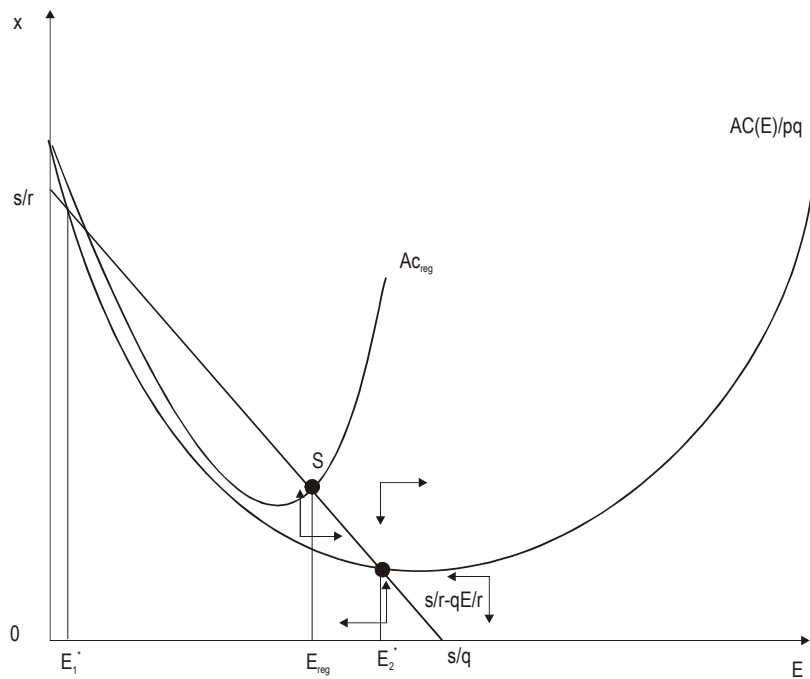Figure 1: Wavefront solution for biomass in open access equilibrium

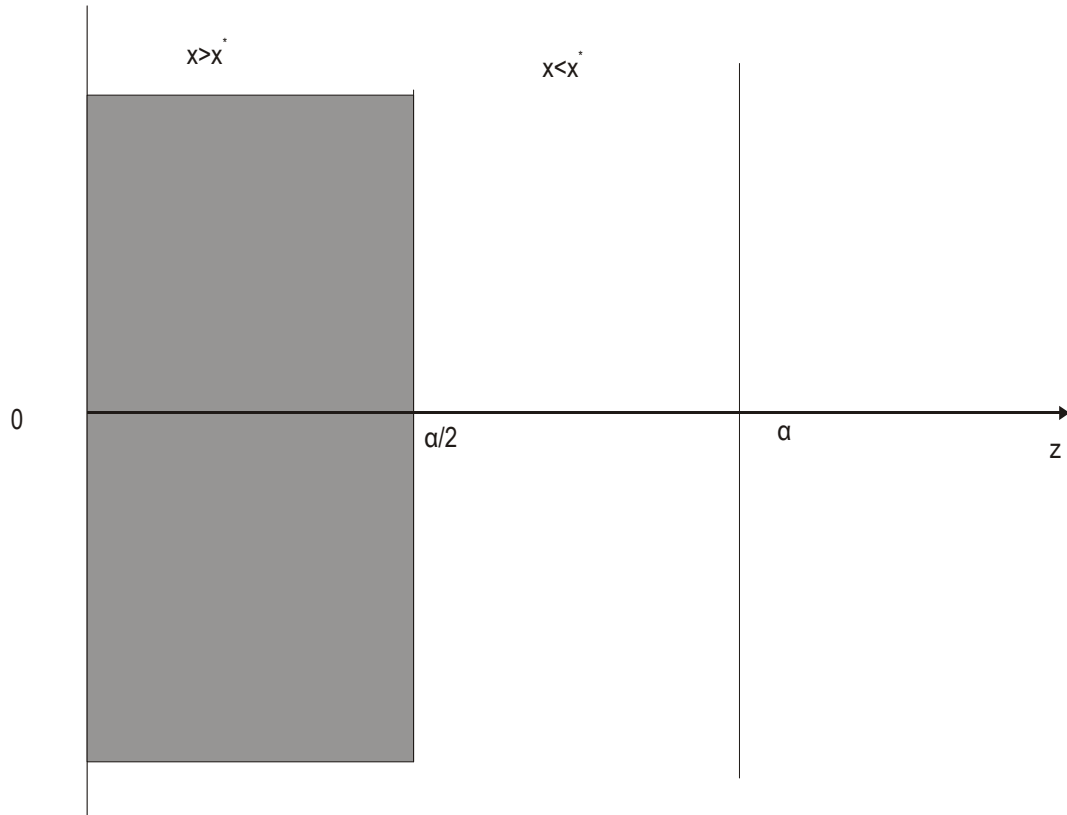Figure 2: Spatially homogeneous solutions and regulation

40

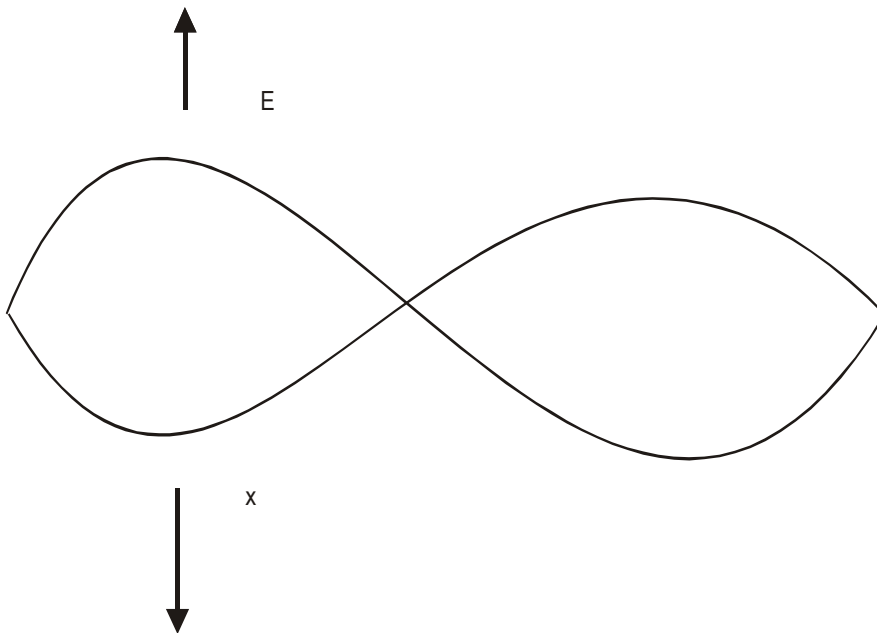Figure 3: A possible pattern formation for the biomass
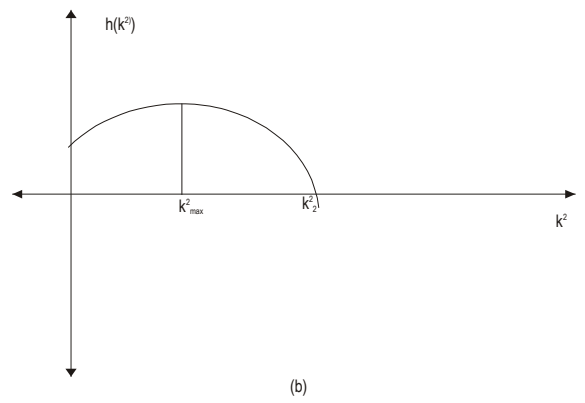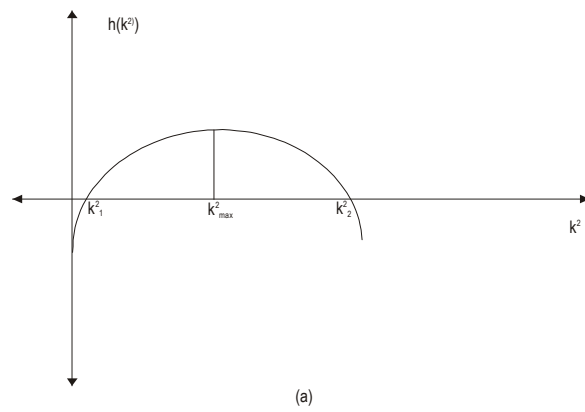

Figure 4: Diffusion driven spatial heterogeneity for biomass and effort

Figure 5: Dispersion relationships