

# Uncertainty Aversion and Robust Portfolio Choices

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Optimal portfolio rules are derived under uncertainty aversion by formulating the portfolio choice problem as a robust control problem. Using a power utility function of the form  $C^\gamma$  with  $0 < \gamma < 1$ , we present the solution of the robust portfolio choice problem in the cases of one and two risky assets. In particular, for two risky assets and one risk-free asset case, we confirm our earlier theoretical result [30], that under uncertainty aversion the total holdings of risky assets as a proportion of the investor's wealth could increase as compared to the holdings under the Merton rule, which is the standard risk aversion case.

*Key Words:* Uncertainty Aversion, Model Misspecification, Robust Control, Portfolio Choice Models.

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## 1. INTRODUCTION

In finance uncertainty has been used to describe the realization of an event for which the true probability distribution is known and thus the expected utility maximization criterion can be used as a methodological framework. Pure uncertainty, where the state space of outcomes is known but the decision maker is unable to assign probabilities, has largely been ignored.

Two main approaches have emerged recently for analyzing the problem of choice when the decision maker faces pure uncertainty in the Knightian sense (or ambiguity) and whose preference relationship is characterized by uncertainty aversion (Gilboa and Schmeidler [12]). In the first, the multiple priors model, the decision maker may formulate his/her objective by attaching a probability, say  $e$ , to a baseline prior and a probability  $(1 - e)$  to the infimum of a family of the disturbed priors around the baseline one. This is the so-called  $e$ -contamination approach (Epstein and Wang [9]), which is consistent with uncertainty or ambiguity aversion.<sup>2</sup> The other, the robust dynamic control approach, models decision making in the presence of model misspecification. In this approach the decision maker is unsure about his/her model, in the sense that there is a group of approximate models that are also considered as possibly true given a set of finite data,

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<sup>2</sup>Chen and Epstein [7] introduce ambiguity aversion to recursive multiple-prior models of utility by considering  $\kappa - Ignorance$  which is a concept that allows differentiation between ambiguous and pure risk cases.

or to put it differently the agent is unsure about what probability measure to use in order to form mathematical expectations. These approximate models are obtained by disturbing a benchmark model, and the admissible disturbances reflect the set of possible probability measures that the decision maker is willing to accept, or to put it differently, how far from the initial 'reference' model the agent is willing to depart. The objective of the robust control approach is to choose a rule that will work well under a variety of model specifications. This methodology provides another tractable way to incorporate uncertainty aversion (e.g. Hansen and Sargent, [19], [20], [21], [23], Hansen et al. [24]).

Portfolio choice theory has been a prominent area of application of the above approaches<sup>3</sup> (e.g. Dow and Werlang [5], Epstein and Wang [9], Chen and Epstein [7], Epstein and Miao [8], Uppal and Wang [29], Maenhout [25], Pathak [26], Liu [13], [14]). The idea behind the use of robust control methods in optimal portfolio choice is that the investor has doubts about the benchmark model and suspects that it is misspecified regarding the assets' price processes. Thus, although the available data

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<sup>3</sup>Monetary policy can be regarded as the initial area of application of these approaches (e.g., Brainard, [1] Hansen and Sargent [23], Onatski and Stock [17], Onatski and Williams [18], Soderstrom [28]). See also Brock and Durlauf [2], Brock, Durlauf and West [3] for similar approaches to policy design and policy evaluation under uncertainty. Another area of interest is environmental and resource management where uncertainty aversion can be used to formulate the concept of the Precautionary Principle (Brock and Xepapadeas [4], Roseta-Palma and Xepapadeas [27])

used to estimate the probability law characterizing the evolution of asset prices allow for the estimation of a benchmark model, there is a set of alternative models describing the evolution of asset prices which is also consistent with the data and could be regarded as possibly true. In this set-up, the investor tries to find a portfolio rule that will work well, in the sense of maximizing utility, under a range of different model specifications of the assets' price equations. When there is no preference for robustness, or to put it differently, there is no concern for model misspecification, then the so-called robustness parameter  $\theta \rightarrow \infty$ .<sup>4, 5</sup>

A central result underlying the recent robust control literature in the portfolio selection context (Maenhout [25], Uppal and Wang [29]) suggests that model uncertainty implies cautiousness in the sense that the investor,

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<sup>4</sup>The robustness parameter  $\theta$  can be interpreted as the Lagrangian multiplier associated with an entropy constraint, which determines the maximum specification error in the asset price equation that the investor is willing to accept (Hansen and Sargent [21]). As such it is a fixed parameter and characterizes preferences consistent with Gilboa and Schmeidler's axiomatization of uncertainty aversion. When  $\theta \rightarrow \infty$  there is no concern about model misspecification and we are in the usual risk aversion framework.

<sup>5</sup>In recent attempts to study the dynamic portfolio rules using robust control methodology, (Maenhout [25], and Uppal and Wang [29]) use certain transformations to eliminate  $\theta$  from the portfolio rule. As shown by Pathak [26] these transformations break down the consistency of preferences with Gilboa and Schmeidler's axiomatization of uncertainty aversion. It seems that since the exogeneity of  $\theta$  is required in order for the problem to be consistent with uncertainty aversion, robust portfolios are parametrized by  $\theta$ . To estimate  $\theta$  in order to fully characterize the robust portfolio, Hansen and Sargent [19] suggest the use of detection error probabilities.

under uncertainty aversion, will invest a smaller share of his/her wealth in the risky assets relative to the share implied by the standard Merton rule under risk aversion. In more general terms, model uncertainty seems to have been associated in the earlier literature with some kind of cautious or conservative behavior,<sup>6</sup> although more recent results in the area of monetary policy analysis under uncertainty seem to provide mixed findings, that is aggressiveness or robustness depending on the structure of the model.<sup>7</sup>

The present paper attempts to derive optimal portfolio rules under uncertainty aversion by following Hansen and Sargent's approach and formulating the portfolio choice problem as a robust control problem. Using a power utility function  $C^\gamma$  with  $0 < \gamma < 1$ , we explicitly derive portfolio rules for the cases of one and two risky assets, allowing for uncertainty aversion, or preference for robustness, with respect to the joint distribution of the assets. Our portfolio rules are parametrized by the robustness parameter  $\theta$  and show that as  $\theta \rightarrow \infty$  the robust portfolio rule tends to Merton's rule in accordance with Maenhout's results.

We present initially the solution of the robust portfolio choice problem for the case of one risky asset case. In this case the robust portfolio rule never leads to an increase in the fraction of our wealth invested in the risky asset relative to the standard risk aversion case, associated with  $\theta \rightarrow \infty$ . We

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<sup>6</sup>For example Brainard's [1] results suggest caution in the face of model uncertainty in a Bayesian framework.

<sup>7</sup>See for example Onatski and Williams [18] and the papers cited by them.

provide numerical calculations for the cases where  $\gamma = 0.5$  and  $\gamma = 0.75$ . Then we derive the optimal robust portfolio rule for the case of two risky assets and present numerical solutions when  $\gamma = 0.5$ , depicting how the fraction of invested wealth in each risky asset and how the overall holdings of risky assets and the optimal consumption rate evolve as a function of the robustness parameter  $\theta$ .

These solutions confirm our earlier theoretical results [30], where under uncertainty aversion, the associated robust portfolio rule indicates that the total holdings of risky assets as a proportion of the investor's wealth is not always smaller as compared to the holdings under the Merton rule (which is the risk aversion case) and which is equivalent to no concerns about model misspecification and  $\theta \rightarrow \infty$ .

This result seems to depart from the belief that uncertainty, or ambiguity aversion, and the associated robust control methods might result in more cautiousness or conservatism regarding portfolio choices, in the sense that holdings of the "risky - ambiguous" assets are reduced relative to the pure risk case.

Finally it should be noted that by parametrizing our robust portfolio rules using the exogenous parameter  $\theta$ , and not eliminating it as Maenhout [25] and Uppal and Wang [29] do, we preserve the consistency of preferences with Gilboa and Schmeidler's axiomatization of uncertainty aversion.

## 2. ROBUST PORTFOLIO CHOICES WITH ONE RISKY ASSET

We consider a market which consists of one riskless asset whose price evolves accordingly to:

$$dS(t) = rS(t)dt \quad S(0) = S_0, \quad t \geq 0,$$

where  $r$  denotes the risk-free rate of return, and one risky asset. Denoting by  $\alpha_1$  the drift rate, or mean rate of return, and by  $\sigma_1$  the volatility rate, the evolution of the prices  $P_1$  of the risky asset is given by the standard geometric Brownian motion:

$$\begin{aligned} dP(t) &= \alpha_1 P(t)dt + \sigma_1 P(t)dB_1(t) \quad t \geq 0, \\ P(0) &= P_0, \end{aligned} \tag{1}$$

where  $\{B_1(t) : t \geq 0\}$  denotes a standard Brownian motion on an underlying probability space  $(\Omega, \mathcal{F}, \mathcal{P})$ . Supposing that  $w_1, w_2$  are the fractions of the wealth,  $W$ , invested in the risky and riskless asset, then:

$$w_1 + w_2 = 1. \tag{2}$$

Therefore the equation for wealth dynamics becomes:

$$dW = w_1(\alpha_1 - r)Wdt + (rW - C)dt + W\sigma_1 w_1 dB_1, \tag{3}$$

where in the above equation  $C$  is the consumption rate. Merton's solution ([15], [16]) of the optimal portfolio allocation problem for an infinite time horizon and one risky asset, determines the optimal portfolio weight,  $w_1$ ,

that is the fraction of the investor's total wealth  $W$  allocated to the risky asset as:

$$U'(C) = V_W, \quad (4)$$

$$w_1 = \frac{(r - \alpha_1) V_W}{\sigma^2 W V_{WW}}, \quad (5)$$

$$\frac{V_W}{V_{WW}} = \frac{U'(C)}{U''(C) \frac{\partial C}{\partial W}}. \quad (6)$$

Following Hansen and Sargent [23], Hansen et al. [24], the above model is regarded as a benchmark model. If the consumer-investor was sure about the benchmark model then there would be no concerns about robustness to model misspecification. Otherwise, concerns for robustness to model misspecification can be reflected by a family of stochastic perturbations, so that the probabilities implied by (1) are distorted. The measure  $\mathcal{P}$  is replaced by another probability measure  $\mathcal{Q}$ . The perturbed model is constructed by replacing  $B_1(t)$  in (1) with

$$B_1(t) = \hat{B}_1(t) + \int_0^t h(s) ds, \quad (7)$$

where  $\{\hat{B}_1(t) : t \geq 0\}$  is a Brownian motion and  $\{h(t) : t \geq 0\}$  is a measurable drift distortion. Therefore using equation (7) the corresponding equation (3), for wealth dynamics, becomes:

$$dW = w_1(\alpha_1 - r + \sigma_1 h)W dt + (rW - C)dt + W\sigma_1 w_1 d\hat{B}_1 \quad (8)$$

As shown in Hansen et al. [24] the discrepancy between the distribution  $\mathcal{P}$  and  $\mathcal{Q}$  is measured as the relative entropy



$$R(\mathcal{Q}) = \int_0^\infty e^{-\delta u} \mathbb{E}_{\mathcal{Q}} \left( \frac{h^2}{2} \right) du \quad (9)$$

and if  $R(\mathcal{Q})$  is finite then

$$\mathcal{Q} \left\{ \int_0^\infty h^2 du < \infty \right\} = 1 \quad (10)$$

and  $\mathcal{Q}$  is locally absolutely continuous with respect to  $\mathcal{P}$ . Under model misspecification, a multiplier robust control problem can be associated with the problem of maximizing the present value of lifetime expected utility, or:

$$\max_{w_1, C} \mathbb{E}_0 \int_0^\infty e^{-\delta t} U(C) dt \quad (11)$$

In this case the multiplier robust control problem becomes:

$$J(\theta) = \sup_{w_1, C} \inf_h \mathbb{E}_{\mathcal{Q}} \int_0^\infty e^{-\delta t} [U(C) + \theta \frac{h^2}{2}] dt \quad (12)$$

subject to (8).

In the above equation  $\theta$  denotes the so-called robustness parameter which takes values greater than or equal to zero. As shown by Hansen and Sargent [22],  $\theta$  is the Lagrangian multiplier at the optimum, associated with the entropy constraint  $\mathcal{Q}(\tau) = \{Q \in \mathcal{Q} : R_t(Q \parallel \mathcal{P}) \leq \tau \ \forall t\}$ . A value of  $\theta = \infty$  indicates that we are absolutely sure about the measure  $\mathcal{P}$ , with no preference for robustness. This case can be regarded as the risk aversion case and the problem is reduced to the standard Merton problem with the objective function given by (11). Lower values for  $\theta$  indicate preference for robustness under model misspecification, or uncertainty aversion, where a value of  $\theta = 0$  indicates that we have no knowledge about the measure  $\mathcal{P}$ .

Using the results of Fleming and Souganidis [10] regarding the existence of a recursive solution to the multiplier problem, Hansen et al. [24] show that problem (12) can be transformed into a stochastic infinite horizon two-player game between the investor and Nature. Nature plays the role of a "mean agent" and chooses a reduction  $h$  in the mean return of assets to reduce the investor's life time utility. The Bellman-Isaacs conditions for this game imply that the value function  $V(W, \theta)$  satisfies the following equation:

$$\delta V = \max_{w_1, C} \min_h \left\{ U(C) + \theta \frac{h^2}{2} + \left( w_1(\alpha_1 - r + \sigma_1 h)W + rW - C \right) V_W + \frac{1}{2} W^2 \sigma_1^2 w_1^2 V_{WW} \right\}. \quad (13)$$

The solution of the above problem is given by the following equations:

$$U'(C) = V_W, \quad (14)$$

$$h = -\frac{\sigma V_W W w_1^*}{\theta}, \quad (15)$$

$$w_1^* = \frac{(r - \alpha_1) V_W}{\sigma^2 W} \frac{1}{V_{WW} \left( 1 - \frac{V_W^2}{\theta V_{WW}} \right)}, \quad (16)$$

$$\frac{V_W}{V_{WW}} = \frac{U'(C)}{U''(C) \frac{\partial C}{\partial W}},$$

where  $w_1^*$  denotes the fraction of the wealth invested in the risky asset when there are concerns about model misspecification and the decision maker tries to find robust decision rules. Furthermore, if we compare  $w_1, w_1^*$  given by (5) and (16), respectively,

$$\frac{w_1}{w_1^*} = 1 - \frac{V_W^2}{\theta V_{WW}} > 1. \quad (17)$$

Thus independently of the utility function and the value of the robustness parameter, concerns for model misspecification decrease the fraction of the wealth invested in the risky asset relative to the standard Merton case. Moreover as  $\theta \rightarrow \infty$  the robust portfolio weight tends to Merton's optimal weight and the utility maximizer acts as if he/she knows the initial benchmark model with certainty.

### 2.1. Power utility function

In order to understand how a preference for robustness influences optimal choices, this section presents the case of robust portfolio rules with a power utility function  $C^\gamma$  with  $0 < \gamma < 1$ . Substituting (14), (15), (16) into (13) and restricting our attention to the class of value functions of the form  $V(W, \theta) = Q W^\gamma$ , we show that the parameter  $Q$  satisfies the following equation:

$$\begin{aligned} \delta = & Q^{\frac{1}{\gamma-1}} - \frac{1}{\theta} \frac{(r - \alpha_1)^2}{\sigma_1^2} \left( \frac{\gamma}{\gamma-1} \right)^2 Q^3 W^\gamma \left( \frac{1}{1 - \frac{\gamma}{\gamma-1} \frac{1}{\theta} Q^2 W^\gamma} \right)^2 \\ & + \gamma (r - Q^{\frac{1}{\gamma-1}}) - \frac{(r - \alpha_1)^2}{\sigma_1^2} \left( \frac{\gamma}{\gamma-1} \right) Q \left( \frac{1}{1 - \frac{\gamma}{\gamma-1} \frac{1}{\theta} Q^2 W^\gamma} \right) \\ & + \frac{1}{2} \frac{(r - \alpha_1)^2}{\sigma_1^2} \frac{\gamma}{\gamma-1} Q \left( \frac{1}{1 - \frac{\gamma}{\gamma-1} \frac{1}{\theta} Q^2 W^\gamma} \right)^2 \end{aligned} \quad (18)$$

For various parameter constellations we are able to calculate, from the above equation, the value of the parameter  $Q$  and, in the sequence using equation (16), to derive the corresponding value of the optimal robust

portfolio weight  $w_1^*$  . More specifically for

$$\delta = 0.05 \quad \alpha_1 = 0.05 \quad \sigma_1 = 0.5 \quad r = 0.03 \quad W = 100$$

the first six figures correspond to the solution for  $\gamma = 0.5$  and  $\gamma = 0.75$ . Particularly they depict the value of the parameter  $Q$ , the optimal robust weight  $w_1^*$  and the optimal consumption rate  $C^*$  as  $\theta$  varies from 0.1 to 200. From these pictures we conclude that an uncertainty averse investor in the first case prefers to invest more in the risky asset and to consume less than in the case where the value of  $\gamma$  is greater. Moreover as the value of  $\theta$  increases, which means that the preference for robustness is declining, the optimal consumption rate decreases and the value of the optimal portfolio weight goes up, in both cases. This implies that as uncertainty aversion is declining the investor increases the holding of the risky asset.

[Figure1]

[Figure2]

[Figure3]

[Figure4]

[Figure5]

[Figure6]

### 3. TWO RISKY ASSETS

In this section we will assume that the market consists of one risk free asset and two correlated risky assets, where  $\rho$  denotes the correlation coefficient at the benchmark model. In this case  $B = [B_1, B_2]^T$  is a vector of independent Brownian processes defined on an underlying probability space  $(\Omega, \mathcal{F})$ , with measure  $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$ . Because of  $\mathbb{E}(dB_1 d\hat{B}_2) = \rho dt$ , where  $\mathbb{E}$  denotes expected value and  $dB_1, d\hat{B}_2$  are correlated Brownian motions on  $\mathcal{P}_1, \mathcal{P}_2$  respectively, the evolution of the prices of the assets can be written as :<sup>8</sup>

$$dP_1(t) = \alpha_1 P_1(t)dt + \sigma_1 P_1(t)dB_1(t) \quad t \geq 0 \quad (19)$$

$$dP_2(t) = \alpha_2 P_2(t)dt + \sigma_2 P_2(t)\rho dB_1(t) + \sigma_2 P_2(t)\sqrt{1 - \rho^2}dB_2(t) \quad (20)$$

Merton's solution for the maximization problem (11) in the two risky assets case is:

$$w_1 W = \frac{A(\alpha_1 - r)\sigma_2^2}{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)} - \frac{A(\alpha_2 - r)\sigma_{12}}{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)} \quad (21)$$

$$w_2 W = -\frac{A(\alpha_1 - r)\sigma_{12}}{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)} + \frac{A(\alpha_2 - r)\sigma_2^2}{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)} \quad (22)$$

$$A = -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C)\frac{\partial C}{\partial W}} \quad (23)$$

Perturbing each Brownian motion separately,<sup>9</sup> the initial measure  $\mathcal{P}$  is replaced by another probability measure  $\mathcal{Q} = \mathcal{Q}_1 \otimes \mathcal{Q}_2$ . At this stage we

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<sup>8</sup>We have that for independent Brownian motions  $B_1, B_2$ :  $\mathbb{E}(dB_1 dB_2) = 0$ ,

$\mathbb{E}(dB_1 dB_1) = dt$  so we write  $d\hat{B}_2 = \rho dB_1 + \sqrt{1 - \rho^2}dB_2$ .

<sup>9</sup>For this reason use the specific form of equations (19), (20).

consider distortions to the joint distribution of assets so we impose an overall entropy constraint for the two assets. Based on Corollary C3.3 of Dupuis and Ellis [6], the entropy constraint becomes:

$$R(\mathcal{Q} \parallel \mathcal{P}) = \sum_{i=1}^2 R(\mathcal{Q}_i \parallel \mathcal{P}_i) = \sum_{i=1}^2 \int_0^\infty e^{-\delta u} \mathbb{E}_{\mathcal{Q}} \left( \frac{h_i^2}{2} \right) du. \quad (24)$$

The above equation allows us to consider two separate distortion terms, one for each asset. However in order to reduce the complexity of the model, we assume symmetric distorted measures  $\mathcal{Q}_i$ , and examine the case with the same distortion terms  $h_i$ . In this specific case, the equation for wealth dynamics becomes:

$$\begin{aligned} dW &= w_1(\alpha_1 - r + \sigma_1 h)W dt + w_2(\alpha_2 - r + \sigma_2(\rho h + h\sqrt{1-\rho^2}))W dt \\ &\quad + (rW - C)dt + W\sigma_1 w_1 d\hat{B}_1 + W\sigma_2 \rho w_2 d\hat{B}_1 + \\ &\quad \sigma_2 \sqrt{1-\rho^2} w_2 d\hat{B}_2. \end{aligned} \quad (25)$$

In this specific case the Bellman equation for problem (12) subject to (25) is:

$$\begin{aligned} \delta V &= \max_{w_i, C} \min_h \left\{ U(C) + \left( w_1(\alpha_1 - r + \sigma_1 h)W + (rW - c) + \theta_2 \frac{h^2}{2} + \right. \right. \\ &\quad \left. \left. w_2(\alpha_2 - r + \sigma_2(\rho + \sqrt{1-\rho^2})h)W \right) V_W + \right. \\ &\quad \left. \frac{1}{2} V_{WW} \sum_{i=1}^2 \sum_{j=1}^2 w_i w_j \sigma_{ij} W^2 \right\}, \end{aligned} \quad (26)$$

where  $\theta_2 = 2\theta$  and  $\theta$  this time refers to the robustness parameter in the two assets case. The first order conditions which describe the solution of

the above two-player game are:

$$U'(C) = V_W \quad (27)$$

$$h = -\frac{V_W W (\sigma_1 w_1^* + \sigma_2 (\rho + \sqrt{1 - \rho^2}) w_2^*)}{\theta_2} \quad (28)$$

$$\sum_{j=1}^2 w_j^* W \sigma_{1j} = A(\alpha_1 - r) + A\sigma_1 h \quad (29)$$

$$\sum_{j=1}^2 w_j^* W \sigma_{2j} = A(\alpha_2 - r) + A\sigma_2 (\rho + \sqrt{1 - \rho^2}) h \quad (30)$$

$$A = -\frac{V_W}{V_{WW}} = -\frac{U'(C)}{U''(C) \frac{\partial C}{\partial W}}. \quad (31)$$

Using matrix notation the solution of the above problem can be described by the following equation:

$$\begin{bmatrix} w_1^* W & w_2^* W \end{bmatrix} \Lambda = \begin{bmatrix} A(\alpha_1 - r) & A(\alpha_2 - r) \end{bmatrix} \quad (32)$$

where:

$$\Lambda = \begin{bmatrix} \sigma_{11} \left(1 - \frac{V_W^2}{\theta_2 V_{WW}}\right) & \sigma_{12} \left(1 - \frac{V_W^2}{\theta_2 V_{WW}} \frac{\rho + \sqrt{1 - \rho^2}}{\rho}\right) \\ \sigma_{12} \left(1 - \frac{V_W^2}{\theta_2 V_{WW}} \frac{\rho + \sqrt{1 - \rho^2}}{\rho}\right) & \sigma_{22} \left(1 - \frac{V_W^2}{\theta_2 V_{WW}} (1 + 2\rho \sqrt{1 - \rho^2})\right) \end{bmatrix}. \quad (33)$$

If  $\Sigma$  denotes the diagonal matrix with elements  $\sigma_1, \sigma_2$  then:

$$\Lambda = \Sigma \begin{bmatrix} \left(1 - \frac{V_W^2}{\theta_2 V_{WW}}\right) & \left(\rho - \frac{V_W^2}{\theta_2 V_{WW}} (\rho + \sqrt{1 - \rho^2})\right) \\ \left(\rho - \frac{V_W^2}{\theta_2 V_{WW}} (\rho + \sqrt{1 - \rho^2})\right) & \left(1 - \frac{V_W^2}{\theta_2 V_{WW}} (1 + 2\rho \sqrt{1 - \rho^2})\right) \end{bmatrix} \Sigma. \quad (34)$$

Solving the above system we determine the fraction of the wealth invested in the first and second asset under robust portfolio choices as:

$$\begin{bmatrix} w_1^*W & w_2^*W \end{bmatrix} = \frac{1}{(1-\rho^2)(1-2\frac{V_W^2}{\theta_2 V_{WW}})} \begin{bmatrix} A(\alpha_1 - r) & A(\alpha_2 - r) \end{bmatrix} \Sigma^{-1} \begin{bmatrix} (1 - \frac{V_W^2}{\theta_2 V_{WW}}(1 + 2\rho\sqrt{1-\rho^2}) & -\rho + \frac{V_W^2}{\theta_2 V_{WW}}(\rho + \sqrt{1-\rho^2}) \\ -\rho + \frac{V_W^2}{\theta_2 V_{WW}}(\rho + \sqrt{1-\rho^2}) & (1 - \frac{V_W^2}{\theta_2 V_{WW}}) \end{bmatrix} \quad (35)$$

In the above equation as  $\theta \rightarrow \infty$  we obtain Merton's solution (21), (22) which corresponds to the standard risk aversion case.

Substituting (35), (28) into (26) and giving initial values to the parameters appearing in equation (26), we are able to determine the value of  $Q$  and afterwards to obtain the optimal robust portfolio weights using the above matrices equation. Assuming that:

$$\begin{aligned} \delta &= 0.05 \quad \alpha_1 = 0.05 \quad \sigma_1 = 0.5 \quad r = 0.03 \\ \alpha_2 &= 0.08 \quad \sigma_2 = 0.5 \quad \rho = 0.7 \quad W = 100 \end{aligned}$$

the next four figures present the solution for  $\gamma = 0.5$ . Figure 7 refers to the optimal portfolio weight invested in the first risky asset,  $w_1^*$ , where the negative sign corresponds to the short selling assumption which is consistent with Merton's model. The next figure corresponds to the second risky asset where we see that as the preference for robustness declines, that is we tend to be satisfied by the benchmark model, the fraction of the wealth,  $w_2^*$ , invested in the risky asset increases. Then we show how the overall holdings



of risky assets  $w_1^* + w_2^*$  change as the value of the robustness parameter increases and afterwards we give the evolution of the optimal consumption  $C^*$  as a function of  $\theta$ . In this specific case we are able to conclude from figure 9, that as the the value of  $\theta$  increases, which is equivalent with the fact that our confidence about the reference models increases, the total fraction invested in the risky assets increases as well.

[Figure7]

[Figure8]

[Figure9]

[Figure10]

In the sequence we repeat the same calculations considering now the following values for our parameters:

$$\begin{aligned} \delta &= 0.05 \quad \alpha_1 = 0.04 \quad \sigma_1 = 0.43 \quad r = 0.03 \\ \alpha_2 &= 0.05 \quad \sigma_2 = 0.87 \quad \rho = 0.93 \quad W = 100 \end{aligned}$$

In this specific case we see that uncertainty aversion induces an increase in the holdings of the one risky asset as compared to risk aversion. When this happens the holding of the other asset is reduced. Moreover we see that the total fraction of the portfolio invested in the two risky assets decreases as the value of the robustness parameter increases, which means that reduction of the ambiguity of our initial benchmark model implies

a reduction in the total holdings of risky assets, a result that is nit in line with the previous examples and the general belief under which model uncertainty has been associated with some kind of conservative behavior.

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[Figure11]

[Figure12]

[Figure13]

[Figure14]

Finally in the last picture considering the following parameter constellation:

$$\begin{aligned} \delta &= 0.05 \quad \alpha_1 = 0.05 \quad \sigma_1 = 0.5 \quad r = 0.03 \\ \alpha_2 &= 0.08 \quad \sigma_2 = 0.8 \quad W = 100 \quad \gamma = 0.5 \end{aligned}$$

we show how the fraction of the wealth allocated to the risky assets changes, as a function both of the correlation coefficient and the value of the robustness parameter  $\theta$ .

[Figure15]

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<sup>10</sup>In our previous work [30], we have proved that this happens whenever the following equation is satisfied:

$$\begin{aligned} \hat{\lambda}(\rho - \sqrt{1 - \rho^2} - \frac{1}{\sigma}) &> \sigma(1 - 2\rho\sqrt{1 - \rho^2}) - (\rho - \sqrt{1 - \rho^2}) \\ \text{with } \hat{\lambda} &= \frac{a_2 - r}{a_1 - r}, \quad \sigma = \frac{\sigma_2}{\sigma_1} \end{aligned}$$

#### 4. CONCLUDING REMARKS

Using a power utility function we explicitly derive robust portfolio rules parametrized by the robustness parameter  $\theta$ , which is not endogenized in order to keep the model consistent with the Gilboa Schmeidler axiomatization of uncertainty aversion.<sup>11</sup> Our solutions confirm the fact that under the robust portfolio rule, the total holdings of risky assets may increase under uncertainty aversion relative to the risk aversion case, which is a result that can be contrasted to results suggesting that robust portfolio choices imply a reduction in the total holdings of risky assets. The fact that changes could go either way depending on the structure of the model parameters suggests that uncertainty aversion and adoption of robust portfolio rules should not be associated with smaller holdings of risky assets.

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<sup>11</sup>Thus the full characterization of the robust portfolio rule requires estimation of  $\theta$ , using for example detection probabilities [23].

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