

Non-Cooperative Exercise Boundaries and Regulation under
Uncertainty: The Case of Cost-Reducing R&D
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Abstract

This paper extends the concept of the exercise boundary as an analytical tool in determining when an optimizing agent might undertake an irreversible action under uncertainty, to situations where the objective function of optimizing agents depends on decisions taken by other agents. By using the case of cost reducing R&D in a fixed numbers oligopoly under demand and technological uncertainty, the exercise boundaries and the corresponding optimal R&D accumulation paths are determined for the non-cooperative, cooperative and socially optimal cases. Comparison of the exercise boundaries makes possible the formulation of R&D policy in the form of subsidies.

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1. Introduction

In the mathematical formulation of problems from physics, biology or economics, the concept of the free boundary emerges as a boundary separating space-time regions with different properties, whose location is not known *a priori*. Free boundaries occur naturally in areas such as material processes, population dynamics, combustion theory reaction-diffusion processes or fluid-flows analysis. In economics free, or as they are sometimes called, exercise boundaries occur in the analysis of investment type problems under uncertainty and irreversibility. This type of problem typically involves capacity expansion (Dixit and Pindyck 1994), development of natural resources (Scheinkman and Zariphopoulou 2001) or investment in financial assets. In economics the properties associated with the regions into which the free boundary separates time-space correspond to the decision to undertake the irreversible decision (e.g. invest) in one region or not to undertake the irreversible decision (e.g. do not invest) in the other region.

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In the problems analyzed so far in economics, the basic underlying assumption is that the boundary is determined by the optimization problem of a single economic agent, which is independent of decisions taken by other agents. This independence assumption does not hold of course when market imperfections or other externalities are introduced. Introducing dependence of an agent's exercise boundary on the actions undertaken by other agents naturally extends the free boundary concept to the realm of a non-cooperative game. This extension is the purpose of the present paper.

The exercise boundary for a firm is derived under demand and technological uncertainty, by considering the problem of optimal accumulation of process R&D associated with small innovations, in a fixed numbers oligopoly with spillovers, where firms behave non-cooperatively both in output and R&D selection. In this problem the exercise boundary associated with the optimization problem of each firm is determined under the assumption that the firm takes as given the actions of its rivals regarding output in the short-run and R&D in the long-run. Thus a non-cooperative exercise boundary, which for each firm depends both on the firm's own accumulated R&D and that of its rivals, is defined.

The non-cooperative exercise boundary is used to analyze the behavior of the firms under demand shocks. The basic response mechanism for cost reducing R&D accumulation implied by this approach allows, when firms are not symmetric, for immediate responses or delayed responses or no responses at all to demand shocks. Convergence to a Nash equilibrium R&D accumulation could be prevented if demand shocks persists, since each demand shock could trigger a shift of the exercise boundary. The shift alters the regions that the non-cooperative exercise boundary separates time-space for each firm and thus changes the structure of its decision space.

A cooperative exercise boundary is determined by assuming that the firms compete in the output selection stage but cooperate in the R&D selection stage by engaging in R&D cartelization (Kamien et al. 1992). Finally, we consider the case of a social planner that maximizes consumer plus producer surplus and derives a socially-optimal exercise boundary.

By comparing the three boundaries we are able to derive an optimal R&D subsidy, optimal in the sense that the the subsidy is chosen so that the noncooperative or the cooperative boundaries coincide with the socially-

optimal boundary. In this way the firms react to any stochastic shock in the same way that the social planner would have reacted.

The present paper contributes to the literature, first by extending the exercise boundary concept to the case of strategic interactions among optimizing agents, and second by applying this methodological framework in order to characterize the process of cost reducing R&D accumulation under demand and technological uncertainty, and describing a policy scheme that corrects deviations between the privately optimal and socially-optimal accumulation paths. An interesting property of the optimal policy scheme in the case of constant cost reducing effects and constant spillovers, is that the optimal subsidy is state independent and deterministic and thus independent of the random variable realization.

2. Non-Cooperative R&D with Spillovers

Consider an industry with $i = 1, \dots, n$ firms $2 \leq n < \infty$. The firms at each instant t face an inverse demand function $p_t = p(\alpha_t, Q_t)$ where $Q_t = \sum_{i=1}^n q_t^i$ is the total quantity produced, q_t^i is the output of firm i , and α_t is the parameter of demand uncertainty. The inverse demand function decreases with Q_t and increases with α_t .

The parameter of demand uncertainty is assumed to follow a geometric Brownian motion process, or:

$$d\alpha_t = y\alpha_t dt + \sigma\alpha_t d\omega_t \quad (1)$$

In (1) $\{\omega_t\}$ is a standard Brownian motion, which represents the source of demand uncertainty, and which is defined on the underlying probability space $\{\Omega, \mathcal{F}, P\}$. The parameters y and σ are constants and represent the mean and the dispersion coefficient of the demand uncertainty parameter. By considering a linear demand or $p_t = \alpha_t - Q_t$, the uncertainty parameter can be interpreted as uncertainty associated with the size of the market.¹ We assume that the market size at each point in time, a_t , takes values in the interval (a_1, a_2) , $a_1, a_2 > 0$, that determines a minimum and a maximum

¹If some long-run market size $\bar{\alpha}$ is assumed, then the movements of the uncertainty parameter can be modelled by a mean reverting process

$$d\alpha_t = y(\bar{\alpha} - \alpha_t) dt + \sigma\alpha_t d\omega_t$$

size. For an exit time τ_a from (a_1, a_2) , $\Pr(\lim_{t \rightarrow \tau_a} a_t = a_j) = 0, j = 1, 2$. Thus a_1 and a_2 are repelling boundaries.

Let x_t^i denote the amount of accumulated R&D in improving production processes undertaken by firm i at time t . Thus accumulated processes R&D in each firm is defined as

$$x_t^i = \int_0^t \Delta x_u^i du, i = 1, 2, \dots, n$$

where $\Delta x_{t'}^i$ is R&D flow undertaken at each time t' . No depreciation is assumed and $\Delta x_{t'}^i \geq 0$ so that the R&D accumulation process is irreversible. The unit production cost of each firm is assumed to be

$$v^i(x_t^i, \mathbf{x}_t^{-i}), i, j = 1, 2, i \neq j$$

where $\mathbf{x}_t^{-i} = (x_t^1, \dots, x_t^{i-1}, x_t^{i+1}, \dots, x_t^n)$ is the vector of accumulated technology improving R&D of the rest of the $j \neq i, j = 1, \dots, n$ rival firms.

It is assumed that the unit production cost of each firm decreases with accumulated R&D in the firm or $\frac{\partial v^i}{\partial x_t^i} < 0$. So technology is assumed to improve gradually over time as a result of accumulated R&D. Thus improvement of production processes is considered to be a consequence of 'routine' environmental R&D (Baumol 1992).

Following D'Aspremont and Jacquemin (1988) or Suzumura (1992), R&D externalities or spillovers are also assumed, so that $\frac{\partial v^i}{\partial x_t^j} \leq 0, j \neq i, i, j = 1, \dots, n$. Therefore the accumulated technology improving R&D in one firm could generate improvements in the unit production of the other firm, without any payment on the part of the firms that experience reductions in unit costs. We make the following additional assumptions regarding the unit cost function

$$\begin{aligned} (i) \quad & 0 < v^i(0, \mathbf{x}^{-i}) < \infty, \lim_{x^i \rightarrow \infty} v^i(x^i, \mathbf{x}^{-i}) = 0^+ \\ (ii) \quad & \lim_{x^i \rightarrow 0^+} \frac{\partial v^i(x^i, \mathbf{x}^{-i})}{\partial x^i} = -M < -\infty, \lim_{x^i \rightarrow \infty} \frac{\partial v^i(x^i, \mathbf{x}^{-i})}{\partial x^i} = 0^- \\ (iii) \quad & D^i = \left(\sum_{j \neq i}^n v^j(x^j, \mathbf{x}^{-j}) - n v^i(x^i, \mathbf{x}^{-i}) \right) < 0 \end{aligned}$$

$$(iv) \quad D_{x^i}^i = \left(\sum_{j \neq i}^n \frac{\partial v^j(x^j, \mathbf{x}^{-j})}{\partial x^i} - n \frac{\partial v^i(x^i, \mathbf{x}^{-i})}{\partial x^i} \right) > 0$$

Assumptions (i) and (ii) impose some regularity on the problem, assumption (iii) implies that unit cost functions are not very different among firms, while assumption (iv) implies that the effects of own R&D on own unit costs are always larger than cross effects. Furthermore for any symmetric vector $\mathbf{x}_t = (x_t^1, \dots, x_t^n)$, $\frac{\partial v^i}{\partial x_t^i} < \frac{\partial v^j}{\partial x_t^j}$, $j \neq i$.

The cost for increasing the stock of accumulated R&D by Δx_t^i is defined as $(1-s)c(\Delta x_t^i)$, where $c(\Delta x_t^i)$ represents strictly convex adjustment costs, with $c'(0) > 0$ and $s \in [0, 1)$ is a subsidy potentially given by the government to cover some of the R&D expenses.

Given this set-up the firm has to decide about output production and R&D expansion. At each instant of time the firm decides about the optimal output level *given* the overall accumulated R&D, and, under Cournot-Nash assumptions, given the output of its rival. Thus output is the operating variable and output decisions can be regarded as “short-run” decisions while R&D decisions are “long-run” decisions.

Short-run profits are defined as:²

$$\pi_t^i = [\alpha_t - Q_t - v^i(x_t^i, \mathbf{x}_t^{-i})] q_t^i$$

Thus the Nash-Cournot short-run equilibrium output for any *given* current level of total R&D is defined as:³

$$q_t^{iN} = \frac{1}{n+1} \left[\alpha_t + \left(\sum_{j \neq i}^n v^j(x_t^j, \mathbf{x}_t^{-j}) - n v^i(x_t^i, \mathbf{x}_t^{-i}) \right) \right] = \frac{a_t + D_t^i}{n+1}, \quad a_t > D_t^i, \quad \forall i \quad (2)$$

where, under symmetry, the non-cooperative short-run equilibrium output becomes

$$q_t^N = \frac{a_t - v(x_t, \mathbf{x}_t^{-i})}{n+1}$$

²It is assumed that $\frac{\partial^2 \pi^i}{\partial q_i \partial q_j} < 0$, $\forall i, j, i \neq j$, so that firms' outputs are strategic substitutes for any given R&D vector \mathbf{x}_t .

³Linear demand and constant short-run marginal costs guarantee the existence and uniqueness of the short run Nash-Cournot equilibrium.

Then the reduced form instantaneous profit function can be defined as:⁴

$$\pi_t^{i*}(\alpha_t, x_t^i, \mathbf{x}_t^{-i}) = (q_t^{iN})^2$$

The short-run Nash-Cournot output equilibrium feeds into the equilibrium for the environmental R&D accumulation which is a long-run problem.⁵ This equilibrium is determined by maximizing the expected discounted profit flow net of the costs of R&D accumulation, given initial conditions $\alpha_0 = \alpha$, $x_0^i = x^i$, $\mathbf{x}_0^{-i} = \mathbf{x}^{-i}$. Since short-run profits for each firm depend on the amount of R&D accumulated by all firms the value function, for each firm also depends on the accumulated R&D of its rivals.

Let $\mathcal{B}^I = \left\{ \Delta x_t^i : \Delta x_t^i \geq 0, \forall t \geq 0 \text{ and } \int_0^t \Delta x_u^i du < \infty, \forall t \geq 0 \right\}$. The set of admissible controls which represent additions in technology improving R&D is defined as $\mathcal{U}_x^i = \left\{ \Delta x_t^i : \Delta x_t^i \in \mathcal{B}^I, \forall x^i \in [0, \infty) \right\}$

The long-run problem for each firm is therefore to choose a non-negative, non-decreasing process of cumulative environmental R&D to solve the following maximization problem:

$$\max_{\mathcal{U}_x^i} J^i(\alpha_t, x_t^i, \mathbf{x}_t^{-i}; \Delta x_t^i), \quad J^i = \mathcal{E}_0 \int_0^\infty e^{-\rho t} [\pi_t^{i*} - (1-s)c(\Delta x_t^i)] dt$$

subject to (1) with $\rho > y$

The value function for this problem is defined as:

$$V^i(\alpha, x^i, \mathbf{x}^{-i}) = \sup_{\mathcal{U}_x^i} J^i(\alpha_t, x_t^i, \mathbf{x}_t^{-i}; \Delta x_t^i)$$

By the concavity of the profit function $\pi_t^{i*} - (1-s)c(\Delta x_t^i)$, in x_t^i and the linearity of the dynamics with respect to the state and the control variables the value function is concave in x_t^i . The dynamic programming equation takes the form (Soner 1997):

$$\rho V^i = \max_{\Delta x^i \geq 0} \left\{ [\mathcal{L}_{\Delta x^i}^{\alpha, \mathbf{x}}] V^i + \pi^{i*} - (1-s)c(\Delta x^i) \right\} \quad (3)$$

⁴It is assumed that the Hessian matrix of the instantaneous profit function π_t^{i*} is negative definite.

⁵See also Dixit (1991) where the short-run equilibrium is determined by maximizing the total flow surplus.

where $\mathcal{L}_{\Delta x^i}^{a, \mathbf{x}}$ is the differential generator

$$\mathcal{L}_{\Delta x^i}^{a, \mathbf{x}} = \frac{1}{2} \sigma^2 \alpha^2 \frac{\partial^2}{\partial \alpha^2} + y \alpha \frac{\partial}{\partial \alpha} + \Delta x^i \frac{\partial}{\partial x^i}$$

Assume that the time line is partitioned into ‘sufficiently small’ time intervals Δt . In each such interval each firm has two choices: to undertake new R&D or not. It is assumed that when the firm decides about which of the two choices to follow, it considers as given the *current* R&D level of its rivals at the beginning of this time interval.⁶ In a sense when $\Delta t \rightarrow 0$ each firm plays best response to its rivals current choice of cumulative R&D. Consider now a time interval $[0, \mathcal{T}_i]$ in which firm i makes no additions to its R&D for the given \mathbf{x}_t^{-i} , $t \in [0, \mathcal{T}_i]$. We define such an interval following Dixit and Pindyck (1994), as a continuation interval or no action interval. A stopping time for firm i is a non-negative random variable \mathcal{T}_i , at which the firm increases its R&D level. Let $x_i^*(\mathcal{T}_i)$ be the optimal R&D process at time \mathcal{T}_i , then following Fleming and Soner (1993) or Soner (1997), if \mathcal{T}_i is a stopping time then by the dynamic programming principle:

$$\begin{aligned} V^i(\alpha, x^i, \mathbf{x}^{-i}) &= \sup_{\mathcal{U}_{\mathbf{x}}^i} \mathcal{E}_0 \int_0^{\mathcal{T}_i} e^{-\rho u} [\pi_t^{i*} - (1-s)c(\Delta x_t^i)] dt + \\ &e^{-\rho \mathcal{T}_i} V^i(\alpha_{\mathcal{T}_i}, x_{\mathcal{T}_i}^i, \mathbf{x}_{\mathcal{T}_i}^{-i}) \end{aligned}$$

Assume now that in some interval $[0, \theta]$, firm i decides not to undertake any new R&D but keep to it at the level x^{i0} given \mathbf{x}_t^{-i0} , $t \in [0, \theta]$, so that $[0, \theta]$ is a no action interval as defined above. Then, the definition of optimal stopping time and the principle of dynamic programming imply that the value function for firm i will be no less from the payoff in the no action interval plus the expected value after θ , or

$$V^i(\alpha, x^i, \mathbf{x}^{-i}) \geq \mathcal{E}_0 \int_0^\theta e^{-\rho u} \pi_t^{i*} dt + e^{-\rho \theta} V^i(\alpha_\theta, x_\theta^i, \mathbf{x}_\theta^{-i}) \quad (4)$$

with equality if it is optimal not to increase R&D in the interval $[0, \theta]$.

⁶Conditioning the decision to undertake or not new R&D on the current accumulated R&D vector (x_t^1, \dots, x_t^n) is a form of a feedback rule (see for example Reinganum 1982).

Applying Itô's lemma to the value function on the right hand side of (4), dividing by θ and taking limits as $\theta \rightarrow 0$, we obtain :

$$\rho V^i \geq \frac{1}{2} \sigma^2 \alpha^2 V_{\alpha\alpha}^i + y \alpha V_{\alpha}^i + \pi_t^{i*} (\alpha_t, x_t^i, \mathbf{x}_t^{-i}) \quad (5)$$

with equality if $x_i(t) = x_i^0$ in the interval $[0, \theta]$.

Consider now the decision to accumulate R&D instantaneously by $\Delta x^i = x^{i0+} - x^{i0}$. Then the right hand side of the dynamic programming equation (3) becomes

$$\begin{aligned} \max_{\Delta x_i \geq 0} \left\{ \left[\mathcal{L}_{\Delta x_i}^{\alpha, \mathbf{x}} \right] V^i + \pi_i^* - (1-s)c(\Delta x^i) \right\} = \\ \frac{1}{2} \sigma^2 \alpha^2 V_{\alpha\alpha}^i + y \alpha V_{\alpha}^i + \pi^{i*} + \widehat{\mathcal{H}}(V_{x^i}^i) \end{aligned}$$

where

$$\widehat{\mathcal{H}}(V_{x^i}^i) = \max_{\Delta x_i \geq 0} \{ V_{x^i}^i \Delta x^i - (1-s)c(\Delta x^i) \}$$

which implies

$$\begin{aligned} V_{x^i}^i - (1-s)c'(\Delta x^i) &\leq 0, \Delta x^i \geq 0, \text{ or} \\ \text{If } V_{x^i}^i - (1-s)c'(\Delta x^i) < 0 &\text{ then } \Delta x^i = 0 \\ \text{If } \Delta x^i > 0 &\text{ then } V_{x^i}^i - (1-s)c'(\Delta x^i) = 0 \end{aligned} \quad (6)$$

Thus when no development is optimal, (5) is satisfied as equality, while when development is optimal, (6) is satisfied. Combining (5) and (6), the Hamilton-Jacobi-Bellman equation is defined as:

$$\min \left\{ \begin{aligned} &[\rho V^i - \frac{1}{2} \sigma^2 \alpha^2 V_{\alpha\alpha}^i - y \alpha V_{\alpha}^i - \pi^{i*}(\alpha, x^i, \mathbf{x}^{-i})], \\ &- [V_{x^i}^i - (1-s)c'(\Delta x^i)] \end{aligned} \right\} = 0 \quad (7)$$

There will be $i = 1, \dots, n$ HJB equations, one for each firm. At every point of the state space $(\alpha, x^i, \mathbf{x}^{-i})$, either the first or the second term of the HJB equation is satisfied as equality while the other is satisfied as inequality. Therefore the state space is divided into two regions: the no-new R&D region (no action interval), and the new R&D region.

3. The Nash Exercise Boundary

The HJB equation (7) determines the conditions under which a firm will undertake new R&D or not given the R&D choice of the other firm. Thus the HJB for firm i divides, for given \mathbf{x}^{-i} , the (α, x^i) space into two regions. The curve $\alpha = \alpha^i(x^i, \mathbf{x}^{-i})$ $i = 1, \dots, n$ for the boundary between the two regions determines the R&D process for each firm. From the one side of the curve, say region I, no new R&D is undertaken. This is the no action space. From the other side, say region II, new R&D is undertaken. We call this curve the Nash exercise boundary (NEB), since it determines the decision whether to undertake new R&D or not *given* the R&D choice of the other firm.

Using the definition of the reduced form instantaneous profit function, the HJB equation in region I implies:

$$\rho V^i - \frac{1}{2}\sigma^2 \alpha^2 V_{\alpha\alpha}^i - y\alpha V_{\alpha}^i = \left[\frac{\alpha + \left(\sum_{j \neq i}^n v^j(x^j, \mathbf{x}^{-j}) - nv^i(x^i, \mathbf{x}^{-i}) \right)}{(n+1)} \right]^2$$

The general solution of this second order differential equation can be obtained as:⁷

$$V^i(\alpha, x^i, \mathbf{x}^{-i}) = A_1^i(x^i, \mathbf{x}^{-i}) \alpha^{\beta_1} + A_2^i(x^i, \mathbf{x}^{-i}) \alpha^{\beta_2} + \Pi(\alpha, x^i, \mathbf{x}^{-i})$$

where $\beta_1 = \frac{1}{2} - \frac{y}{\sigma^2} + \sqrt{\left(\frac{y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1$ is the positive root, and β_2 is the corresponding negative root of the fundamental quadratic $Q = \frac{1}{2}\sigma^2\beta(\beta - 1) + y\beta - \rho = 0$, and $\Pi(\alpha, x^i, \mathbf{x}^{-i}; s)$ is the particular solution. We need to disregard the negative root in order to prevent the value from becoming infinitely large when the market size becomes very small, thus we set $A_2^i(x^i, \mathbf{x}^{-i}) = 0$.⁸ Using the method of undetermined coefficients we obtain the particular solution as:

$$\begin{aligned} \Pi(\alpha, x^i, \mathbf{x}^{-i}) &= \Pi_0(x^i, \mathbf{x}^{-i}) + \alpha\Pi_1(x^i, \mathbf{x}^{-i}) + \alpha^2\Pi_2 \\ \Pi_0(x^i, \mathbf{x}^{-i}) &= \frac{\left(\sum_{j \neq i}^n v^j - nv^i\right)^2}{\rho(n+1)^2} = \frac{[D^i]^2}{\rho(n+1)^2} \end{aligned}$$

⁷The homogeneous part of this differential equation is an Euler equation.

⁸See Dixit and Pindyck (1994).

$$\begin{aligned}\Pi_1(x^i, \mathbf{x}^{-i}) &= \frac{2\left(\sum_{j \neq i}^n v^j - nv^i\right)}{(n+1)^2(\rho-y)} = \frac{2D^i}{(n+1)^2(\rho-y)} \\ \Pi_2 &= \frac{1}{(n+1)^2(\rho-\sigma^2-2y)}\end{aligned}$$

Thus the solution becomes:

$$V^i(\alpha, x^i, \mathbf{x}^{-i}) = A_1^i(x^i, \mathbf{x}^{-i}) \alpha^{\beta_1} + \Pi(x^i, \mathbf{x}^{-i}) \quad (8)$$

In region II the second term of the HJB is zero and $\Delta x_i > 0$ or,

$$V_{x^i}^i(\alpha, x^i, \mathbf{x}^{-i}) - (1-s)c'(\Delta x^i) = 0 \quad (9)$$

The NEB $\alpha = \alpha^i(x^i, \mathbf{x}^{-i})$ can be obtained by solving (8) for α . The unknown functions $A_1^i(x^i, \mathbf{x}^{-i})$ and $\alpha = \alpha^i(x^i, \mathbf{x}^{-i})$ can be specified using the ‘value matching’ and the ‘smooth pasting’ conditions.⁹

The value matching condition means that on the boundary separating the two regions, the two value functions should be equal. Then we have, combining (8) and (9), that

$$\begin{aligned}V_{x^i}^i(\alpha, x^i, \mathbf{x}^{-i}) &= \frac{\partial A_1^i(x^i, \mathbf{x}^{-i})}{\partial x^i} \alpha^{\beta_1} + \frac{\partial \Pi_0(x^i, \mathbf{x}^{-i})}{\partial x^i} + \alpha \frac{\partial \Pi_1(x^i, \mathbf{x}^{-i})}{\partial x^i} \\ &= (1-s)c'(\Delta x^i), \quad \alpha = \alpha^i(x^i, \mathbf{x}^{-i})\end{aligned} \quad (10)$$

The smooth pasting or high contact condition means that the derivatives of the value functions with respect to α on the boundary are equal, or:

$$\begin{aligned}V_{\alpha x^i}^i(\alpha, x^i, \mathbf{x}^{-i}) &= \beta_1 \frac{\partial A_1^i(x^i, \mathbf{x}^{-i})}{\partial x^i} \alpha^{\beta_1-1} + \frac{\partial \Pi_1(x^i, \mathbf{x}^{-i})}{\partial x^i} = 0 \quad (11) \\ \alpha &= \alpha^i(x^i, \mathbf{x}^{-i})\end{aligned}$$

Combining (10) and (11) we can solve for the unknown NEB function $\alpha^i(x^i, \mathbf{x}^{-i})$ to obtain

$$\alpha^i(x^i, \mathbf{x}^{-i}) = \frac{\beta_1}{\beta_1-1} \frac{(1-s)c'(\Delta x^i) - \frac{\partial \Pi_0(x^i, \mathbf{x}^{-i})}{\partial x^i}}{\frac{\partial \Pi_1(x^i, \mathbf{x}^{-i})}{\partial x^i}} \quad (12)$$

⁹For a presentation of these conditions see Dixit and Pindyck (1994).

After substituting for Π_0 and Π_1 the NEB for firm $i = 1, \dots, n$ can be written as or

$$\begin{aligned} \alpha^i(x^i, \mathbf{x}^{-i}) &= \frac{\beta_1}{\beta_1 - 1} \left\{ \frac{(1-s)c'(\Delta x^i) - \kappa_1 D^i D_{x^i}^i}{\kappa_2 D_{x^i}^i} \right\} \\ \kappa_1 &= \frac{2}{\rho(n+1)^2}, \quad \kappa_2 = \frac{2}{(\rho-y)(n+1)^2} > 0 \end{aligned} \quad (13)$$

Since $D^i < 0$, $D_{x^i}^i > 0$, the boundary function takes only positive values. Under symmetry the NEB becomes:¹⁰

$$\alpha(x) = \frac{\beta_1}{\beta_1 - 1} \left\{ \frac{(1-s)c'(\Delta x) + \kappa_1 v [(n-1)v'_{-i} - nv'_i]}{\kappa_2 [(n-1)v'_{-i} - nv'_i]} \right\} \quad (14)$$

with $v'_{-i} < 0$ denoting the spillover effect, $v'_i < 0$ denoting the own R&D effect on costs and $(n-1)v'_{-i} - nv'_i = v_x > 0$ by assumption (iv).¹¹

Furthermore after substituting for the NEB in (11) we obtain

$$\frac{\partial A_1^i(x^i, \mathbf{x}^{-i})}{\partial x^i} = -\frac{\kappa_2 D_{x^i}^i}{\beta_1} [\alpha^i(x^i, \mathbf{x}^{-i})]^{1-\beta_1} < 0 \quad (15)$$

or, under symmetry,

$$\frac{\partial A(x)}{\partial x} = -\frac{\kappa_2 v_x}{\beta_1} [\alpha(x)]^{1-\beta_1}, \quad v_x = (n-1)v'_{-i} - nv'_i$$

From the solution (8) for the value function, the particular solution $\Pi(\alpha, x^i, \mathbf{x}^{-i})$ can be interpreted as the expected profits when R&D is kept constant at its initial level, given the R&D level of the rivals. The term $A_1^i(x^i, \mathbf{x}^{-i})$ can be interpreted as the current value of the option to expand R&D in the future given the present level of rivals' R&D. Then $\frac{\partial A_1^i(x^i, \mathbf{x}^{-i})}{\partial x^i}$ is to be interpreted as the marginal option value of R&D expansion given the R&D of the rivals. Since when new R&D is undertaken the firm gives up an option, the marginal option value is negative.

¹⁰We denote $v'_i = \frac{\partial v^i}{\partial x^i} < 0$, $v'_{-i} = \frac{\partial v^i}{\partial x^j} < 0$, $i \neq j$, $v''_{ij} = \frac{\partial^2 v^i}{\partial x^i \partial x^j} \forall i, j$.

¹¹For example if we put $v(X)$, $X = x_i + \sum_{j \neq i} \delta x_j$, then $v'_i = -1$ and $v'_{-i} = -\delta$, $\delta \in (0, 1)$.

In order to explore the structure and the properties of the NEB we use the assumptions made about the unit cost functions. From (i) - (iii) it follows that

$$\alpha^i(x^i, \mathbf{x}^{-i}) > 0, \forall \mathbf{x}, \alpha^i(0, \mathbf{x}^{-i}) < \infty, \lim_{x^i \rightarrow \infty} \alpha^i(x^i, \mathbf{x}^{-i}) = \infty$$

Furthermore

$$\begin{aligned} \frac{\partial \alpha^i(x^i, \mathbf{x}^{-i})}{\partial x^i} = & \left(\frac{\beta_1}{\beta_1 - 1} \right) \frac{1}{(\kappa_2 D_{\mathbf{x}^i}^i)^2} \left\{ \kappa_2 \left[(1-s)c'' - \kappa_1 \left([D_{\mathbf{x}^i}^i]^2 + D^i D_{\mathbf{x}^i \mathbf{x}^i}^i \right) \right] D_{\mathbf{x}^i} - \right. \\ & \left. \left[(1-s)c' - \kappa_1 D^i D_{\mathbf{x}^i} \right] \kappa_2 D_{\mathbf{x}^i \mathbf{x}^i}^i \right\} \\ \text{where } D_{\mathbf{x}^i \mathbf{x}^i}^i = & \left(\sum_{j \neq i}^n \frac{\partial^2 v^j}{\partial x^{2i}} - n \frac{\partial^2 v^i}{\partial x^{2i}} \right) \end{aligned}$$

For the symmetric case we have

$$\begin{aligned} \alpha'(x) = & \frac{\beta_1}{\beta_1 - 1} \frac{1}{(\kappa_2 v_x)^2} \left\{ \kappa_2 \left[(1-s)c'' + \kappa_1 \left((v_x)^2 + v v_{xx} \right) \right] (v_x) - \right. \\ & \left. \kappa_2 \left[(1-s)c' + \kappa_1 v v_x \right] (v_{xx}) \right\}, v_{xx} = (n-1)v''_{-i} - n v''_i \end{aligned}$$

These results imply that the monotonicity of α^i with respect to x^i depends to a large extent on the sign of the term $D_{\mathbf{x}^i \mathbf{x}^i}^i$ which reflects diminishing or increasing returns in cost savings due to R&D expansion. In this context, by abusing notation, $v_{x^i x^j}^i = \frac{\partial^2 v^i}{\partial x^{2j}} < 0$, $i, j = 1, \dots, n$ indicates increasing returns while $v_{x^i x^j}^i > 0$ indicates diminishing returns. Let $v_{x^i x^j}^i < 0$ for $x^i \in \mathcal{X}_1 \subset [0, \infty)$ then under the assumption that direct second-order effects are larger than second-order spillover effects $D_{\mathbf{x}^i \mathbf{x}^i}^i > 0$. Let $v_{x^i x^i}^i > 0$ for $x^i \in \mathcal{X}_2 \subset [0, \infty)$ again under the assumption that direct second-order effects are larger than second-order spillover effects $D_{\mathbf{x}^i \mathbf{x}^i}^i < 0$. Thus during the R&D process there could be intervals of increasing and decreasing returns. It is more plausible to assume that the increasing returns prevail at relatively low R&D levels while at higher levels diminishing returns take over. Assuming

therefore the coexistence of increasing and diminishing returns the following proposition can be stated:

Proposition: (i) Assume $D_{\mathbf{x}^i \mathbf{x}^i}^i < 0$ and $(1-s)c'' > \kappa_1 \left([D_{\mathbf{x}^i}]^2 + D^i D_{\mathbf{x}^i \mathbf{x}^i}^i \right)$ (or $v_{xx} < 0$ and $(1-s)c'' > \kappa_1 \left[(v_x)^2 - vv_{xx} \right]$ for the symmetric case), which is satisfied when adjustment costs are strongly convex, the subsidy is not very high and diminishing returns prevail in the process of cost reduction through R&D. Then the slope of the NEB for firm i is positive. (ii) Assume $D_{\mathbf{x}^i \mathbf{x}^i}^i > 0$ and $(1-s)c'' < \kappa_1 (D_{\mathbf{x}^i}^2 + DD_{\mathbf{x}^i \mathbf{x}^i}^i)$ (or $v_{xx} > 0$ and $(1-s)c'' < \kappa_1 \left[(v_x)^2 - vv_{xx} \right]$ for the symmetric case), which is satisfied when adjustment costs are not strongly convex, the subsidy is relatively high and increasing returns prevail in the process of cost reduction through R&D. Then the slope of the NEB for firm i is negative.

The above results imply that the shape of the NEB could be as shown in figure 1 for two firms i and j , where increasing returns for firm j prevail up to \bar{x}_i .

[Figure 1]

Suppose that $\alpha = \alpha_0$, the two firms have accumulated x_0^j and x_0^i respectively, and α jumps to α_1 . There are different patterns of responses depending on the characteristics of the two firms. Suppose that as in figure 1 the jump crosses the boundary for firm i but not for firm j . Firm i undertakes a large amount of R&D investment, because increasing returns make possible lumpy expansion in R&D,¹² while firm j does not undertake any new investment. The change however in x^i will shift the NEB of the rival firm j .

To analyze these shifts we consider the general case The shift of the NEB of the i th firm from a change in the R&D of its j th rival is determined by the derivative $\frac{\partial \alpha^i(x^i, \mathbf{x}^{-i})}{\partial x^j}$. By noting that $D^i < 0$ this derivative can be written as:

$$\frac{\partial \alpha^i(x^i, \mathbf{x}^{-i})}{\partial x^j} = \frac{1}{(\kappa_2 D_{x^i}^i)^2} \left\{ \kappa_1 \kappa_2 (D_{x^j}^i D_{x^i}^i + D^i D_{x^i x^j}^i) D_{x^i}^i - \right.$$

¹²This result is in accordance with the results obtained by Dixit and Pindyck (1994) for capacity expansion under increasing returns.

$$\kappa_2 D_{x^i x^j}^i \left[(1-s)c'' - \kappa_1 D_{x^i}^i D_{x^i}^i \right] \Big\}$$

$$\text{where } D_{x^j}^i = \left(\sum_{h \neq i} \frac{\partial v^h}{\partial x^j} - n \frac{\partial v^i}{\partial x^j} \right), \quad D_{x^i x^j}^i = \left(\sum_{h \neq i} \frac{\partial^2 v^h}{\partial x^j \partial x^i} - n \frac{\partial^2 v^i}{\partial x^j \partial x^i} \right)$$

In the above expression the terms $\frac{\partial v^i}{\partial x^j}$, $i \neq j$ indicate the spillover parameters, while the second-order terms indicate spillover effects on marginal spillovers. To simplify the expression assume that second-order terms are negligible or $D_{x^i x^j}^i \approx 0$, then

$$\frac{\partial \alpha^i(x^i, \mathbf{x}^{-i})}{\partial x^j} = \frac{\kappa_1}{\kappa_2} D_{x^j}^i = \frac{\rho - y}{\rho} D_{x^j}^i$$

Since $\frac{\partial v^i}{\partial x^j} < 0 \forall i, j$, if the spillover from firm j to firm i is sufficiently strong so that $n \left| \frac{\partial v^i}{\partial x^j} \right| > \left| \sum_{h \neq i} \frac{\partial v^h}{\partial x^j} \right|$ then $\frac{\partial \alpha^i}{\partial x^j} > 0$. Thus an increase in the R&D of firm j will shift upwards the exercise boundary for firm i , and will reduce the likelihood of this firm undertaking new R&D in the presence of upwards demand shocks. Under low spillover from firm j to firm i so that $n \left| \frac{\partial v^i}{\partial x^j} \right| < \left| \sum_{h \neq i} \frac{\partial v^h}{\partial x^j} \right|$, then $\frac{\partial \alpha^i}{\partial x^j} < 0$ and the increase in firm j R&D will shift the boundary downwards for firm i and will increase the likelihood of new R&D.

Under symmetry we obtain $\frac{\partial \alpha^i}{\partial x^j} = \frac{\rho - y}{\rho} (v'_j - 2v'_{-j})$.¹³ Again high spillovers will reduce the likelihood of new R&D. Since under symmetry all firms behave the same, a positive derivative $\frac{\partial \alpha^i}{\partial x^j}$ under high spillovers will indicate underinvestment effects. The opposite holds under low spillovers.

In terms of figure 1, assume that the spillover effects are weak so that the boundary for firm j shifts downwards as a result of i 's R&D and the shift is sufficient so that new investment is undertaken. This new investment will in turn shift downwards the boundary of firm i and will induce new investment. If there are no significant jumps in α and we assume approximately constant spillovers, the shifts will be reducing in size since $\frac{\rho - y}{\rho} < 1$ and there will be convergence to some Nash equilibrium values for x_j, x_i . The process however can be interrupted by a new jump of α , in which case a new process will start.

¹³ $v'_j, v'_{-j} < 0$

The basic adjustment mechanism described here indicates that when firms are not symmetric there could be immediate responses or delayed responses or no responses at all to demand shocks. In general the delayed response takes place when the shock is not sufficiently strong to induce immediate investment by a given firm but investment is a response of this firm to the actions taken by the rivals under weak spillovers. No response to a positive demand shock is possible if, again the shock is not sufficiently strong to induce immediate investment from the firm, but the spillovers effects, because of the rivals' investment are sufficiently strong.

4. Cooperative R&D

Assume that firms behave non-cooperatively in the short-run when they choose output but cooperate in the choice of the long-run R&D variables.¹⁴ Thus R&D is chosen cooperatively as if a group manager would determine the optimal R&D process for each firm in such a way that the expected present value of the sum of the profit flow of each firm is maximized. This is the case called R&D cartelization by Kamien et al. (1992).

The optimal time problem for the cooperative solution can be defined by considering the group manager that faces a given accumulated R&D profile \mathbf{x}_t^0 and considers whether or not it is optimal to have a partial increment Δx^i given \mathbf{x}_t^{-i0} . In analyzing this choice the group manager takes into account the effect of undertaking or not the partial increment Δx^i on the expected profit flow of *all* the firms in the group. It is clear that this effect is not taken into account in the non-cooperative case regarding the R&D choice.

The maximization problem for the cooperative solution is defined as:

$$\max_{\mathcal{U}_x^i} J^{ic}(\alpha_t, x_t^i, \mathbf{x}_t^{-i}; \Delta x_t^i), \quad J^i = \mathcal{E}_0 \int_0^\infty e^{-\rho t} \sum_{i=1}^n [\pi_t^{i*} - (1-s)c(\Delta x_t^i)] dt$$

subject to (1), with the value function defined as:

$$V^{ic}(\alpha, x^i, \mathbf{x}^{-i}) = \sup_{\mathcal{U}_x^i} J^{ic}(\alpha_t, x_t^i, \mathbf{x}_t^{-i}; \Delta x_t^i)$$

¹⁴This assumption is common in the cooperative R&D literature see D'Aspremont and Jacquemin (1992) or Suzumura (1992).

Optimal stopping implies that, when it is optimal for the group for the i th firm's R&D to remain constant in the interval $[0, \theta]$ with the rest of the group's R&D at the level \mathbf{x}_t^{-i0} , then the value function is defined as

$$V^{ic}(\alpha, x^i, \mathbf{x}^{-i}) = \mathcal{E}_0 \int_0^\theta e^{-\rho u} \sum_{i=1}^n [\pi_t^{i*}] dt + e^{-\rho\theta} V^{ic}(\alpha_\theta, x_\theta^i, \mathbf{x}_\theta^{-i})$$

On the other hand when the decision is to accumulate R&D instantaneously for the i th firm the dynamic programming principle implies

$$\rho V^{ic} = \max_{\Delta x_i \geq 0} \left\{ \left[\mathcal{L}_{\Delta x_i}^{a, \mathbf{x}} \right] V^{ic} + \sum_{i=1}^n [\pi_i^* - (1-s)c(\Delta x^i)] \right\}$$

Following the same procedures as in the case of the noncooperative problem, the HJB equation for the cooperative problem can be written as:

$$\min \left\{ \begin{aligned} & [\rho V^{ic} - \frac{1}{2} \sigma^2 \alpha^2 V_{\alpha\alpha}^{ic} - y \alpha V_\alpha^{ic} - \sum_{i=1}^n \pi^{i*}(\alpha, x^i, \mathbf{x}^{-i})], \\ & - [V_{x^i}^i - (1-s)c'(\Delta x^i)] \end{aligned} \right\} = 0$$

The solution for the value function takes the form

$$V^{ic}(\alpha, x^i, \mathbf{x}^{-i}) = A_1^{ic}(x^i, \mathbf{x}^{-i}) \alpha^{\beta_1} + \Pi^c(x^i, \mathbf{x}^{-i})$$

where

$$\begin{aligned} \Pi^c(\alpha, x^i, \mathbf{x}^{-i}) &= \Pi_0^c(x^i, \mathbf{x}^{-i}) + \alpha \Pi_1^c(x^i, \mathbf{x}^{-i}) + \alpha^2 \Pi_2^c \\ \Pi_0^c(x^i, \mathbf{x}^{-i}) &= \frac{\sum_{i=1}^n [D^i]^2}{\rho(n+1)^2} \\ \Pi_1^c(x^i, \mathbf{x}^{-i}) &= \frac{2 \sum_{i=1}^n D^i}{(n+1)^2(\rho-y)} \\ \Pi_2^c &= \frac{n}{(n+1)^2(\rho-\sigma^2-2y)} \end{aligned}$$

Using the value matching and the smooth pasting conditions the cooperative exercise boundary becomes:

$$\alpha^{ic}(x^i, \mathbf{x}^{-i}) = \frac{\beta_1}{\beta_1 - 1} \times \left\{ \frac{(1-s)c'(\Delta x^i) - \kappa_1 \left(D^i D_{x^i}^i + \sum_{i \neq j}^n D^j D_{x^i}^j \right)}{\kappa_2 \left(D_{x^i}^i + \sum_{i \neq j}^n D_{x^i}^j \right)} \right\} \quad (16)$$

The terms $\sum_{i \neq j}^n D^j D_{x^i}^j$ and $\sum_{i \neq j}^n D_{x^i}^j$ are, in the terminology of Kamien et al. (1992), the combined-profits externality generated by the R&D undertaken by firm i on the profits of all other firms. The distinction with the certainty case is that while under certainty this combined-profits externality is realized directly on the firm's profits, in our case the effects are realized through their effects on the cooperative exercise boundary. Under symmetry the cooperative exercise boundary becomes:

$$\begin{aligned} \alpha^c(x) &= \frac{\beta_1}{\beta_1 - 1} \times \\ &\left\{ \frac{(1-s)c'(\Delta x) + \kappa_1 v \left[(n-1)v'_{-i} - nv'_i + (n-1)(v'_i - 2v'_{-i}) \right]}{\kappa_2 \left[(n-1)v'_{-i} - nv'_i + (n-1)(v'_i - 2v'_{-i}) \right]} \right\} \\ &= \frac{\beta_1}{\beta_1 - 1} \left\{ \frac{(1-s)c'(\Delta x) - \kappa_1 v \left[v'_i + (n-1)v'_{-i} \right]}{-\kappa_2 \left[v'_i + (n-1)v'_{-i} \right]} \right\} \end{aligned} \quad (17)$$

The interpretation of the cooperative exercise boundary is similar to the noncooperative case. Comparison of (16) and (17) with (13) and (14) respectively, indicates that there are differences in the relative positions of the boundaries. The relative position of the two boundaries can be determined by the difference: $\alpha^i(x^i, \mathbf{x}^{-i}) - \alpha^{ic}(x^i, \mathbf{x}^{-i})$, and reflects whether R&D expansion will be faster or slower under non-cooperative or cooperative behavior. To obtain a clearer picture of this comparison we examine the symmetric case, for which:

$$\begin{aligned} \alpha(x) - \alpha^c(x) &= -\frac{\beta_1}{(\beta_1 - 1)} \times \\ &\frac{(s-1)c'(\Delta x^i)(n-1)(v'_i - 2v'_{-i})}{\kappa_2 \left[n(v'_i - v'_{-i}) + v'_{-i} \right] \left[v'_i + (n-1)v'_{-i} \right]} \end{aligned}$$

In this expression the denominator is always positive, so if the spillover effects are sufficiently strong so that $|2v'_{-i}| > |v'_i|$, or $(v'_i - 2v'_{-i}) > 0$, then in the symmetric case the non-cooperative boundary lies uniformly above the cooperative boundary. With the non-cooperative boundary uniformly above the cooperative one, as in figure 1, the implication is that when the demand conditions indicate that R&D expansion is desirable, the

expansion is higher under cooperation than under non cooperation. Under cooperation, for a demand shock to α_1 , R&D investment equals $x_0^i x_1^{ic}$ which exceeds the non-cooperative investment $x_0^i x_1^i$.¹⁵ For low spillovers there is the possibility that the cooperative boundary is above the noncooperative one and expansion could be higher without cooperation.

Assume that the spillover effects are not constant at different levels of x and let $\phi^i(x_t^i, \mathbf{x}_t^{-i}) = v_i' - 2v_{-i}'$, $\forall i$. If a vector \mathbf{x}^0 exists such that $\phi^i(\mathbf{x}_t^0) = 0$ then the two boundaries have a crossing point. A continuity argument suggests that from the one side of the crossing point cooperation leads to faster R&D expansion, while from the other side of the crossing point noncooperation leads to faster R&D expansion.

5. The socially-optimal Exercise Boundary

When we consider the case of a social planner that maximizes consumer plus producer surplus we need to distinguish between the non-symmetric and the symmetric case.

In the non-symmetric case the assumption of constant marginal cost and zero fixed costs implies that it will be optimal to cover market demand with only one firm which is the one with the lowest marginal cost, and charge a price equal to this minimum marginal cost. In the symmetric case however it is optimal to have all n firms in the market, each one producing the same output. We examine this case first.

In the symmetric cases the short-run social surplus is defined, for fixed x , as:

$$S_t = \int_0^{Q_t} (a_t - Z_t) dZ_t - \sum_{i=1}^n v^i(x_t^i, \mathbf{x}_t^{-i}) q_t^i, \quad Q_t = \sum_{i=1}^n q_t^i, \quad q_t^i = q_t, \quad v^i = v$$

where the subsidy $s = 0$ for the social optimization problem. The first-order condition for the maximization of the short-run social surplus is the usual price equals marginal cost rule or

$$a_t - Q_t = v(x_t^i, \mathbf{x}_t^{-i})$$

¹⁵This result is similar to the one obtained by D'Aspremont and Jacquemin (1988).

By symmetry the socially-optimal short-run output *given* the accumulated R&D level is defined as:

$$q_t^s = \frac{a_t - v(x_t, \mathbf{x}_t^{-i})}{n}$$

By the definition of the social surplus we have:

$$S_t^* = \left(\sum_{i=1}^n q_t^i \right) \left(\alpha_t - \frac{1}{2} \left(\sum_{i=1}^n q_t^i \right) - v^i \right)$$

Using the first-order conditions the optimal short-run social surplus can be defined under symmetry as:

$$S_t^* = \frac{1}{2} \left(\sum_{i=1}^n q_t^s \right)^2 \quad (18)$$

The R&D choice for the social optimization problem is defined as:

$$\max_{\mathcal{U}_{\mathbf{x}}} J^s(\alpha_t, \mathbf{x}_t; \Delta x_t^i), \quad J^s = \mathcal{E}_0 \int_0^\infty e^{-\rho t} \left[S_t^* - \sum_{i=1}^n c(\Delta x_t^i) \right] dt$$

subject to (1), with the value function defined as:

$$V^s(\alpha, x^i, \mathbf{x}^{-i}) = \sup_{\mathcal{U}_{\mathbf{x}}} J^s(\alpha_t, \mathbf{x}_t; \Delta x_t^i)$$

The HJB equation for the symmetric social optimization problem becomes

$$\min \left\{ \begin{array}{l} [\rho V^s - \frac{1}{2} \sigma^2 \alpha^2 V_{\alpha\alpha}^s - y \alpha V_{\alpha}^s - S^*(\alpha, \mathbf{x})], \\ - [V_x^s - c'(\Delta x)] \end{array} \right\} = 0$$

The solution for the value function takes the form

$$V^s(\alpha, \mathbf{x}) = A_1^s(\mathbf{x}) \alpha^{\beta_1} + \Pi^s(\mathbf{x})$$

where

$$\begin{aligned} \Pi(\alpha, \mathbf{x}) &= \Pi_0^s(\mathbf{x}) + \alpha \Pi_1^s(\mathbf{x}) + \alpha^2 \Pi_2^s \\ \Pi_0^s(\mathbf{x}) &= \frac{\sum_i v^2}{2\rho n^2}, \quad \Pi_1^s(\mathbf{x}) = \frac{-\sum_i v}{(\rho - y)n^2}, \quad \Pi_2^s = \frac{1}{2(\rho - \sigma^2 - 2y)n^2} \end{aligned}$$

Using the value matching and the smooth pasting conditions the socially-optimal exercise boundary becomes:

$$\alpha^s(x) = \frac{\beta_1}{\beta_1 - 1} \left\{ \frac{c'(\Delta x) - \frac{v}{\rho n^2} [v'_i + (n-1)v'_{-i}]}{-\frac{1}{(\rho-y)n^2} [v'_i + (n-1)v'_{-i}]} \right\} \quad (19)$$

We compare again the socially-optimal exercise boundary to the Nash and the cooperative boundaries, for $s = 0$, to obtain:

$$\begin{aligned} a(x) - a^s(x) &= \frac{\beta_1 c'(\Delta x)}{2(\beta_1 - 1)} \times \\ &\frac{(-1 + n(-2 + n(2n - 1)))v'_i + v'_{-i} + n(1 + n - 3n^2)v'_{-i}(\rho - y)}{(n(v'_i - v'_{-i}) + v'_{-i})(v'_i + (n-1)v'_{-i})} \\ a^c(x) - a^s(x) &= \frac{\beta_1}{2(\beta_1 - 1)} \frac{c'(\Delta x)(-1 + (n-2)n)(\rho - y)}{(v'_i + (n-1)v'_{-i})} \end{aligned}$$

In comparing the non-cooperative and the socially-optimal case it is not clear whether the non-cooperative exercise boundary is uniformly above or below the socially-optimal exercise boundary. This suggests that non-cooperative outcomes could imply either insufficient R&D or R&D over-expansion. For the cooperative case it can be seen that for $n > 3$ then $a^c(x) - a^s(x) > 0$, indicating that cooperative R&D accumulation is socially excessive.¹⁶

6. Optimal R&D policy

The comparison of the different types of exercise boundaries indicates that the same demand shock is likely to create different responses depending on the behavioral assumption. A policy question arising from this observation is whether a regulator can induce firms to behave according to the socially-optimal rule.

¹⁶When the firms are non-symmetric only the minimum marginal cost firm produces and the pricing rule implies then that short-run output is determined as:

$$q_t^{sm} = a_t - v^m(x)$$

The social surplus is defined by (18) and the socially optimal exercise boundary for the single firm is given by (19).

It is clear that a subsidy s can be used for such an instrument. The optimal subsidy s^* should be chosen such that

$$a(x; s^*) = a^s(x) \text{ or } a^c(x; s^*) = a^s(x)$$

In this case non-cooperating firms at the R&D stage or cooperating firms at the R&D stage will be induced to follow the socially-optimal rule. By using (14) and (19) we obtain:

$$s_i^* = \frac{1}{(n+1)^2} \left\{ 1 + n \left[2 + 3n - \frac{2n(1+n)v'_i}{v'_i + (n-1)v'_{-i}} \right] \right\}$$

This is a state dependent subsidy that depends on the strength of the own R&D cost reduction effort as well as the strength of the spillover parameter. Since, as it was pointed out in the previous section, there is the possibility of insufficient R&D expansion, or overexpansion at the non-cooperative solution relative to the social optimum, the subsidy could be negative. This is the case of an R&D tax to prevent overexpansion. This result can be made clearer by considering the special case where $v'_i = -1$ and $v'_{-i} = -\delta$, $\delta \in [0, 1)$. In this case the optimal subsidy is state independent and is defined as:

$$s^* = \frac{1}{(n+1)^2} \left\{ 1 + n \left[2 + 3n - \frac{2n(1+n)}{1 + (n-1)\delta} \right] \right\}$$

The subsidy as a function of n and δ is shown in figure 2.

[Figure 2]

For the case of inducing the socially-optimal behavior when firms cooperate at the R&D stage the price instrument becomes:

$$s^{*c} = \frac{1 - (n-2)n}{(n+1)^2}$$

This is a state independent instrument, for $n > 3$ then $s^{*c} < 0$, indicating that because of overexpansion a tax is needed to induce the socially-optimal behavior.

One advantage of designing an exercise boundary policy, especially in the case of constant cost reducing own effects and constant spillovers, is

that the resulting price instrument is state independent and deterministic. Thus instead of defining instruments which are contingent on the realization of a random variable, we define instruments which are independent of the random variable realization. Since the firms respond to movements in the random variables by adjusting their R&D according to their exercise boundary, and the exercise boundary they use has become identical to the socially-optimal exercise boundary by the policy instrument, the deterministic instrument induces the socially-optimal behavior under uncertainty.

7. Simultaneous demand and technological uncertainty

We consider now the possibility of a more general problem than the one analyzed in the previous sections, by allowing for the possibility of stochastic effects on unit production costs. These effects could be interpreted as reflecting stochastic effects on the innovation process or stochastic operating conditions or stochastic failures in technology. To model these stochastic effects we define unit production costs as

$$\tilde{v}_t^i = \theta_t^i v^i(x_t^i, \mathbf{x}_t^{-i})$$

where θ_t^i follows a geometric Brownian motion

$$d\theta_t^i = \eta_t^i \theta_t^i dt + \omega_t^i \theta_t^i dz_{\theta_t^i}^i, \quad \theta_0^i = \theta^i > 0$$

We assume that θ_t^i takes values in the interval (θ_1, θ_2) , that is, $\theta_t^i \in (\theta_1, \theta_2)$, $\forall t$ with $\theta_1, \theta_2 > 0$ and that for an exit time τ_θ from (θ_1, θ_2) , $\Pr(\lim_{t \rightarrow \tau_\theta} \theta_t^i = \theta_j) = 0, j = 1, 2$. Thus θ_1 and θ_2 are repelling boundaries.

To simplify the exposition we further assume that θ_t^i and α_t are uncorrelated, and that θ_t^i is the same for all firms so that $\theta_t^i = \theta_t$ for all i .

The differential generator for the non-cooperative case becomes:

$$\mathcal{L}_{\Delta x^i}^{\alpha, \theta, \mathbf{x}} = \frac{1}{2} \sigma^2 \alpha^2 \frac{\partial^2}{\partial \alpha^2} + \frac{1}{2} \omega^2 \theta^2 \frac{\partial^2}{\partial \theta^2} + y \alpha \frac{\partial}{\partial \alpha} + \eta \theta \frac{\partial}{\partial \theta} + \Delta x^i \frac{\partial}{\partial x^i}$$

and the HJB equation takes the form

$$\min \left\{ \begin{array}{l} [\rho V^i - \frac{1}{2} \sigma^2 \alpha^2 V_{\alpha\alpha}^i - \frac{1}{2} \omega^2 \theta^2 V_{\theta\theta}^i - y \alpha V_\alpha^i - \eta \theta V_\theta^i - \pi^{i*}(\alpha, \theta, x^i, \mathbf{x}^{-i})], \\ - [V_{x^i}^i - (1-s)c'(\Delta x^i)] \end{array} \right\} \quad (20)$$

In the no action region the value function is determined by the partial differential equation (PDE)

$$LV^i = -\pi^{i*}(\alpha, \theta, x^i, \mathbf{x}^{-i}) \quad (21)$$

where L is the operator

$$L = \frac{1}{2}\sigma^2\alpha^2 + \frac{1}{2}\omega^2\theta^2 + y\alpha + \eta\theta - \rho \quad (22)$$

The assumption of repelling boundaries for α and θ implies that the coefficients of the second-order partial derivatives of the PDE (21) are strictly positive. Thus the PDE (21) is uniformly elliptic and the operator L is uniformly elliptic with smooth coefficients. Since $\pi^{i*}(\alpha, \theta, x^i, \mathbf{x}^{-i})$ is also a smooth function it follows from the Weyl lemma¹⁷ that the solution $V^i(\alpha, \theta, x^i, \mathbf{x}^{-i})$ for the value function is a smooth function.

The smoothness of the value function allows the generalization of the NEB to higher dimensions. Let $V^i(\alpha, \theta, x^i, \mathbf{x}^{-i})$ be a smooth solution for (21), then the value matching condition implies that:

$$V_{x^i}^i = (1-s)c'(\Delta x^i) \quad (23)$$

while the high contact condition implies:

$$V_{\alpha x^i}^i = 0 \quad (24)$$

$$V_{\theta x^i}^i = 0 \quad (25)$$

Conditions (23), (24) and (25) can be used to define a smooth surface in the space (α, θ, x^i) . This surface is the NEB. The construction of the NEB can be illustrated by considering a continuum of homogeneous solutions of the form $C_j a^\beta \theta^\gamma$ for the homogeneous part of the PDE, $LV^i = 0$. By direct substitution it can be verified that a solution can take the form:

$$V^i = C_1(x^i, \mathbf{x}^{-i}) a^{\beta_1} \theta^{\gamma_1} + C_2(x^i, \mathbf{x}^{-i}) a^{\beta_2} \theta^{\gamma_2} + \Phi(\alpha, \theta, x^i, \mathbf{x}^{-i}) \quad (26)$$

where (β_j, γ_j) $j = 1, 2$ are defined by the relationship $\frac{1}{2}\sigma^2\beta(\beta-1) + \frac{1}{2}\omega^2\gamma(\gamma-1) + y\beta + \eta\gamma - \rho = 0$. The positive root of the this quadratic in β is:

$$\beta(\gamma) = \frac{1}{2} - \frac{y}{\sigma^2} + \sqrt{\left(\frac{y}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2} + \frac{\gamma}{\sigma^2}(\omega^2(1-\gamma) - 2\eta)}$$

¹⁷**The Weyl Lemma** (Shubin 1987): *If the operator L of the equation $LV^i = -\pi^{i*}$ is elliptic and $\pi^{i*} \in C^\infty(\Theta)$, that is π^{i*} is a smooth function, where Θ is a domain in \mathcal{R}^m , then $V \in C^\infty(\Theta)$.*

By choosing $\gamma_j \in \left(0, 1 - \frac{2\eta}{\omega^2}\right)$, $j = 1, 2$ we obtain $\beta_j = \beta_j(\gamma_j) > 1$. Furthermore, $\Phi(a, \theta, x^i, \mathbf{x}^{-i})$ is a particular solution defined as:

$$\begin{aligned}\Phi &= \xi a \theta + \xi_a a^2 + \xi_\theta \theta^2 \\ \xi &= \frac{2D^i}{(n+1)^2(\rho - y - n)} \\ \xi_a &= \frac{1}{(n+1)^2(\rho - \sigma^2 - 2y)} \\ \xi_\theta &= \frac{1}{(n+1)^2(\rho - \omega^2 - 2\eta)}\end{aligned}$$

Using this class of solution the value matching condition implies:

$$\frac{\partial C_1}{\partial x^i} a^{\beta_1} \theta^{\gamma_1} + \frac{\partial C_2}{\partial x^i} a^{\beta_2} \theta^{\gamma_2} + \frac{\partial \Phi}{\partial x^i} = (1-s)c'(\Delta x^i)$$

while the high contact conditions imply:

$$\beta_1 \frac{\partial C_1}{\partial x^i} a^{\beta_1-1} \theta^{\gamma_1} + \beta_2 \frac{\partial C_2}{\partial x^i} a^{\beta_2-1} \theta^{\gamma_2} + \frac{\partial \Phi}{\partial a} = 0 \quad (27)$$

$$\gamma_1 \frac{\partial C_1}{\partial x^i} a^{\beta_1} \theta^{\gamma_1-1} + \gamma_2 \frac{\partial C_2}{\partial x^i} a^{\beta_2} \theta^{\gamma_2-1} + \frac{\partial \Phi}{\partial \theta} = 0 \quad (28)$$

For $\beta_1 \gamma_2 \neq \beta_2 \gamma_1$ the linear system in $\frac{\partial C_j}{\partial x^i}$ $j = 1, 2$ of (27) and (28) has a unique solution:

$$\begin{aligned}h_1(a, \theta, x^i, \mathbf{x}^{-i}) &= \frac{\partial C_1}{\partial x^i} = \frac{1}{\Omega} a^{\beta_2-1} \theta^{\gamma_2-1} \left[\gamma_2 \frac{\partial \Phi}{\partial a} a - \beta_2 \frac{\partial \Phi}{\partial \theta} \theta \right] \\ h_a(a, \theta, x^i, \mathbf{x}^{-i}) &= \frac{\partial C_2}{\partial x^i} = \frac{1}{\Omega} a^{\beta_1-1} \theta^{\gamma_1-1} \left[\beta_1 \frac{\partial \Phi}{\partial \theta} \theta - \gamma_1 \frac{\partial \Phi}{\partial a} a \right] \\ \Omega &= a^{\beta_1+\beta_2-1} \gamma^{\gamma_1+\gamma_2-1} (\beta_1 \gamma_2 - \beta_2 \gamma_1)\end{aligned}$$

After substituting into the value matching condition we obtain:

$$\begin{aligned}\Psi(a, \theta, x^i, \mathbf{x}^{-i}) &= h_1(a, \theta, x^i, \mathbf{x}^{-i}) a^{\beta_1} \theta^{\gamma_1} + h_2(a, \theta, x^i, \mathbf{x}^{-i}) a^{\beta_2} \theta^{\gamma_2} \\ &+ \frac{\partial \Phi}{\partial x^i} = (1-s)c'(\Delta x^i)\end{aligned} \quad (29)$$

For $\frac{\partial \Psi}{\partial x^i} \neq (1-s)c''(\Delta x^i)$ a solution to (29) exists. Thus the NEB for the three dimensional problem is defined as:

$$x^i = B^i(a, \theta, \mathbf{x}^{-i}), \quad i = 1, \dots, n$$

The NEB is a smooth surface like the one presented in figure 3.

[Figure 3]

Assume that the no new R&D region is above the surface S while the expansion region is below S . For any initial point (x_0^i, a_0, θ_0) the movements of the stochastic variables (a, θ) can be represented by movements around point A on plane E . If for example (a_0, θ_0) moves to F , new investment $\Delta x^i = x_1^i - x_0^i$ is undertaken to restore equilibrium at the boundary.

By analogy to the one dimensional case, the NEB for the case of cooperative R&D can be obtained by the solution for the value function, V^{ic} , in the no action region of the PDE

$$LV^{ic} = - \sum_i \pi^i (\alpha, \theta, x^i, \mathbf{x}^{-i})$$

and the value matching and high contact conditions (23)-(25) where V^i has been replaced by V^{ic} .

On the other hand the socially-optimal exercise boundary is obtained by the solution for the value function, V^s , in the no action region of the PDE

$$LV^s = -S^* (\alpha, \theta, \mathbf{x})$$

and the corresponding value matching and high contact conditions.

Let $x = B(\alpha, \theta; s)$, $x^c = B^c(\alpha, \theta; s)$, $x^s = B^s(\alpha, \theta)$ the exercise boundaries for the non-cooperative, cooperative and socially-optimal cases under symmetry. Then the optimal subsidy can be defined by the solutions:

$$\begin{aligned} s^* & : B(\alpha, \theta; s) = B^s(\alpha, \theta) \\ s^{*c} & : B^c(\alpha, \theta; s) = B^s(\alpha, \theta) \end{aligned}$$

It is clear that the conceptual framework can be extended to higher dimensional problems, although it will undoubtedly be harder to obtain analytical solutions.

8. Concluding Remarks

This paper extends the concept of the exercise boundary as an analytical tool in determining when an optimizing agent might undertake an irreversible action under uncertainty, to situations where the objective function of the optimizing agent depends on decisions taken by other agents. By using the case of cost reducing R&D in a fixed numbers oligopoly under demand and technological uncertainty, the exercise boundary concept for a single firm is naturally extended to non-cooperative and cooperative frameworks. The non-cooperative framework allows for the derivation of an exercise boundary characterized by the Nash assumption, where each firm undertakes irreversible R&D expansion given the R&D expansion choices of its rivals.

The determination of cooperative, non-cooperative and socially-optimal exercise boundaries makes possible the formulation of R&D policy in the form of subsidies. The major advantage of designing regulation in terms of the exercise boundaries relative to the regulatory schemes under uncertainty, is that the policy instrument under constant spillovers, which is the most often analyzed case in the literature, is not contingent upon the realization of a stochastic variable, so that a deterministic instrument can be used under conditions of uncertainty.

Although the cost reducing R&D case was used as a vehicle for exposing the concept of cooperative and non-cooperative exercise boundaries, the approach can be applied to other cases where the irreversible decision of an optimizing agent with respect to a state variable under uncertainty affects the objective function of other agents.

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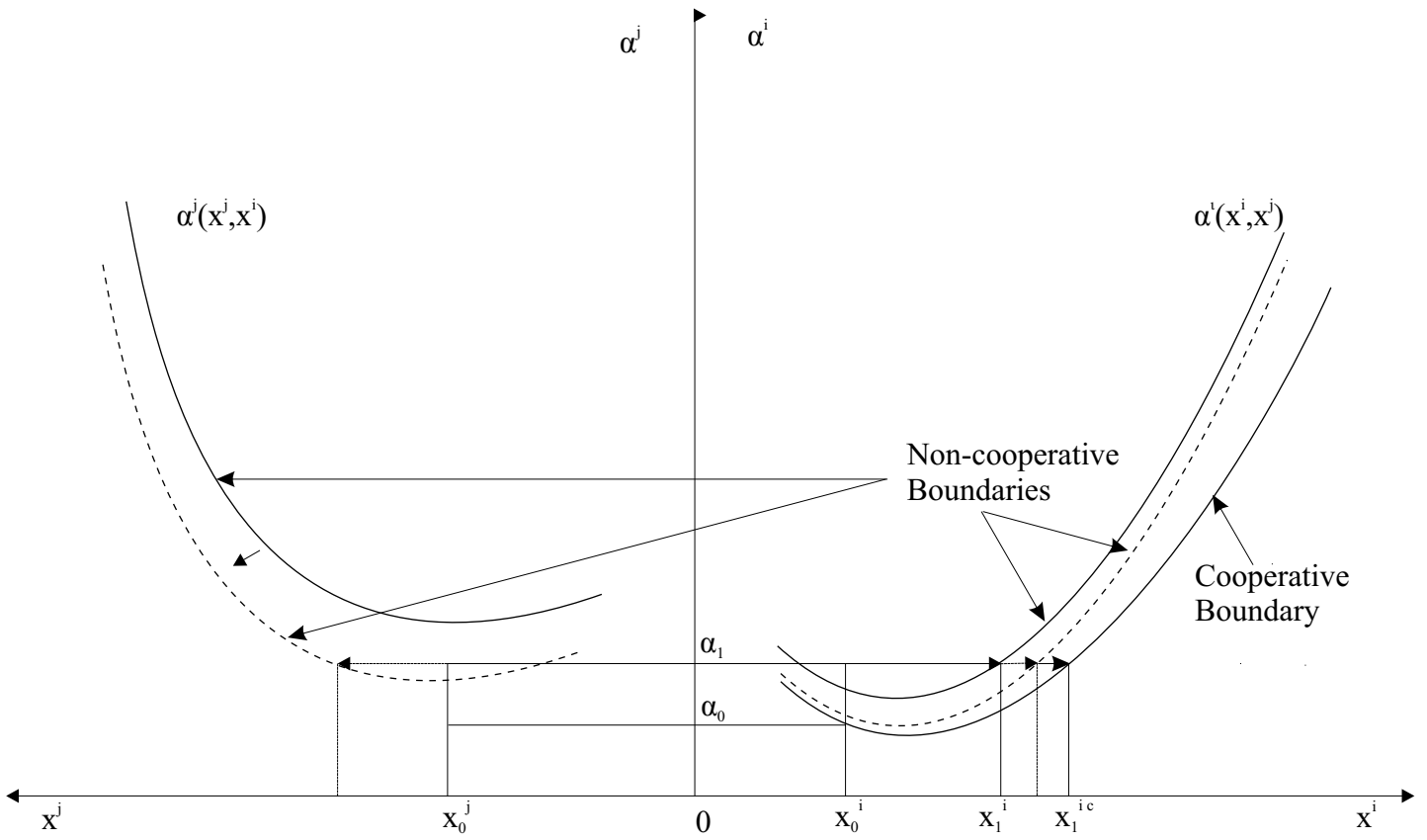


Figure 1: The Nash Exercise Boundary

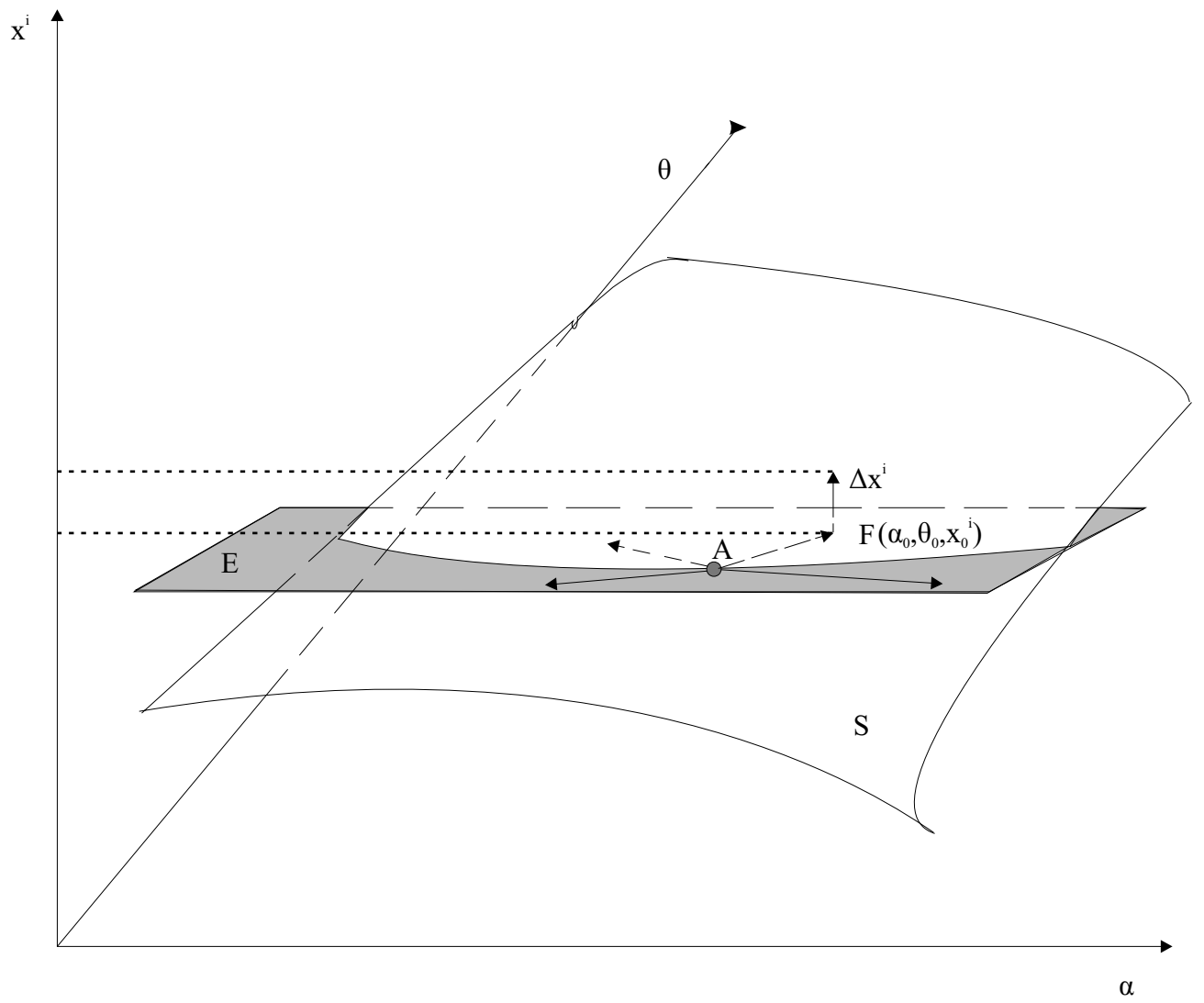


Figure 3: The NEB under market and unit cost uncertainty

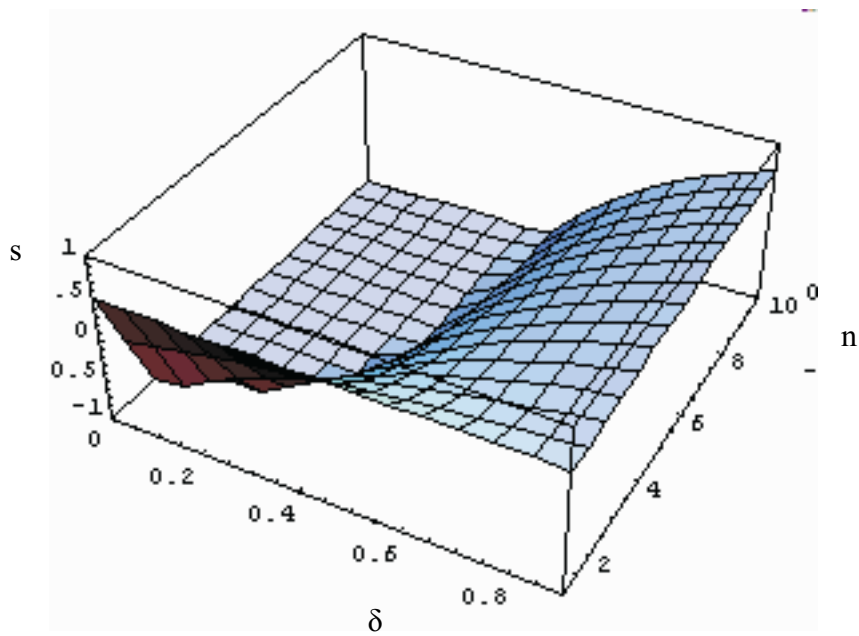


Figure 2: Optimal R&D subsidy with constant spillovers