Strategic capacity investment with common ownership or cross holdings

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Abstract

We study how overlapping ownership affects the timing and size of capacity investments in duopoly. In addition to standard accommodation and delay strategies, internalization allows a leader to block follower entry. Follower timing and capacity reactions are less aggressive, making outcomes less competitive ex-post. Positional competition is more intense, and entry occurs earlier in equilibrium. Internalization raises a leader's incentive to delay follower entry rather than accommodate, and we show with an example that this strategic shift can benefit consumers.

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1 Introduction

Horizontal ownership concentration, where a small number of institutional investors hold significant minority stakes in competing firms, is an increasingly pervasive phenomenon which has raised regulatory concern because of its potential to foster anticompetitive behavior (Backus et al. 2021, Posner et al. 2016). Horizontal minority shareholding, where firms take non-controlling stakes in product market rivals, has also been a growing concern since the turn of the millennium. Such forms of overlapping ownership have attracted a wave of academic interest, kindled notably by evidence of pricing distortions in the airline industry due to common ownership (Azar et al. 2018). New forms of evidence such as natural and laboratory experiments continue to emerge (Heim et al. 2022, Hariskos et al. 2022), altogether lending broad credence to the thesis that managers account for ownership structure in their decision-making by internalizing some of the effects they exert on rival firms.

In the discussion surrounding common ownership, the causal mechanism linking owners to the managerial decisions that determine product market outcomes is a central theme (Hemphill and Kahan 2019, Anton et al. 2021). Institutional investors regularly engage with the management of their portfolio firms (Shekita 2022), and chief among the strategic decisions that top management makes is the exercise of a firm's real options (Smit and Trigeorgis 2017). So far the study of strategic effects of common ownership has centered mostly around R&D investments, but in many of the industries concerned by common ownership firms hold other options, like the option to build production capacity, which are equally important. We propose therefore to examine how increased internalization modifies the timing and size of irreversible capacity investments under uncertainty, and ultimately find a variety of effects whose product-market consequences can be either anti or pro-competitive.

Our model extends the framework for strategic capacity investment in Huisman and Kort (2015) to incorporate overlapping ownership. We thus study two firms holding competing projects in a market which evolves over time. These firms have ownership structures which overlap (either because of common shareholders or because of cross-holdings) so their management internalizes rival value when making investment decisions. Other than this, the firms determine when and how much capacity to install in the standard way. In equilibrium, one of the firms acts as a leader and invests first, whereas the second firm is a follower and reacts to the leader's timing and capacity decision.

We find first of all that internalization invariably exerts an anticompetitive effect on the follower firm, in contrast with prior work involving fixed-size investments where the follower entered earlier if product market profits were very sensitive to internalization (Zormpas and Ruble 2021). In the present model, the weight attributed to leader profit effectively magnifies its capacity from the follower's perspective, driving the follower to enter at a higher demand threshold and to choose a smaller capacity upon entry.

The follower's less aggressive timing and quantity reactions benefit the leader firm, which enjoys a protracted monopoly period followed by less intense duopoly competition. With internalization the leader has the novel possibility of blocking the follower entirely, though it ultimately prefers to either just delay the follower's entry or to accommodate it. The leader prefers to delay (deter) the follower by investing in relatively larger capacities than it otherwise would at low demand states, but shifts to accommodation at higher demand states at which delaying the follower this way becomes prohibitively expensive. Because the leader benefits from both less aggressive capacity and timing reactions if it induces delay strategically, internalization strengthens leader preference for deterrence relative to accommodation. Overall we find that the leader's optimal investment behavior resembles the case without internalization if the level of internalization is not too high, but also that qualitative differences otherwise arise, e.g. accommodation can fail to ever be optimal with sufficient internalization.

If firms compete for the leadership role, we find that the follower's less aggressive reaction makes leading relatively more attractive. Internalization therefore has a procompetitive effect on entry timing in preemption equilibrium. Entry occurs at a low enough demand state that the leader chooses a capacity which delays follower entry. The procompetitive effect of internalization on entry timing is offset however by a lower leader capacity. Introducing endogenous capacities therefore leads us to nuance the results obtained in prior work with fixed investment size.

Beyond these effects at the preemption threshold, we also find that if leader investment occurs at an intermediate demand state, a moderate degree of internalization can exert a procompetitive effect through an altogether different channel pertaining to the leader's strategy choice. Specifically the leader's increased preference for delay with internalization can push it to shift from accommodation to deterrence. Delaying the follower requires a sharp increase in capacity, and we find that this capacity increase can be substantial enough that consumers ultimately benefit, despite the detrimental effect on follower entry.

We complement our analytical results with a numerical analysis which bears out these insights, i.e. that internalization is anticompetitive for the follower but also has a procompetitive effect on the timing of investment in preemption equilibrium. We show moreover that the second procompetitive effect mentioned above due to procompetitive shift in the leader's strategy effectively occurs, and that the resulting increase in capacity can be large enough that overall consumer surplus increases.

To illustrate the idea of a strategic shift which arises in our dynamic model, it is useful to start with the standard Stackelberg-Spence-Dixit model with linear demand $Q = 1 - P$, zero production cost, and a fixed cost of entry $f = .0025$. The firms choose their capacities sequentially. At this fixed cost, the first-mover ordinarily prefers to accommodate follower entry. Suppose however that there is symmetric cross-ownership: each firm i holds an s percent stake in the rival so that, up to a normalization, it maximizes $\pi_i + .01s\pi_{-i}$, with $s \in \{0, 10, 20\}$. The resulting equilibrium leader, follower, and total capacities are:

Table 1: Procompetitive strategic shift (static)

	Q_L^*	$Q_F^*(Q_L^*)$	Q^{Total}
no cross-ownership	.5	.25	.75
10% cross-ownership	.82	0	.82
20% cross-ownership	.75	0	.75

Table 1 shows that a small degree of ownership overlap can have a procompetitive effect on total output and hence consumer surplus (which here is positively related to total output). This is because the overlap (mutual 10% stakes in the rival) induces the leader to adopt a deterrence strategy which increases its output, and this output increase is large enough to outweigh the absence of follower entry and output. With more overlap (mutual 20% stakes), internalization still drives the leader to shift to deterrence, but the effect on total output is no longer procompetitive. In Appendix A.1, we characterize the parameter region over which overlapping ownership causes a procompetitive strategic shift in this model, and further below in Section 6, we return to the idea of a strategic shift in the context of our dynamic model and derive an analogous numerical example involving both follower capacity and timing (see Table 3).

Our study contributes to the growing literature on the effect of overlapping ownership on strategic behavior. Innovation is a key dimension of business strategy, and overlapping ownership has been shown to have a positive effect on investment and welfare in the presence of R&D spillovers (Vives 2020), in innovation contests (Stenbacka and Van Moer 2022), and to facilitate welfare-enhancing technology transfers (Papadopoulos et al. 2019). Our model complements this stream by showing that procompetitive effects arise even more broadly, in industries which are less R&D-intensive. Another research stream addresses how internalization affects Stackelberg leadership, and finds that it facilitates entry deterrence and may raise efficiency (Li et al. 2015, Ma and Zeng 2021). Few authors to our knowledge have studied how the insights regarding overlapping ownership and product market outcomes originally developed in a static setting (Reynolds and Snapp 1986) extend to strategic decisions in a stochastic, dynamic market.

This paper also contributes to the literature on strategic investment with timing and capacity choice (Huisman and Kort 2015) by complementing other studies which have allowed for pre-existing capacities or introduced time-to-build considerations (Huberts et al. 2019, Jeon 2021). The dimension of internalization which we add extends the space of strategies available to the leader firm so as to encompass three possibilities, accommodation, strategic delay, and blockade, that mirror early work on this topic (Dixit 1980).

Section 2 below states the main assumptions of our model. Section 3 studies the follower problem. Section 4 studies the leader's capacity choice and derives its reduced form payoff. Section 5 describes equilibrium investment. In Section 6 we discuss welfare and report a numerical analysis illustrating possible procompetitive effects such as the leader's shift toward delay.

2 Model

An industry consists of two firms which are initially inactive. Their ownership structures are symmetric and overlap. Up to a normalization, each firm maximizes a perceived value

$$\Omega_i = V_i + \lambda V_{-i},\ \lambda \in [0, 1] \tag{1}$$

where V_i denotes the value of its own assets and V_{-i} denotes the value of its rival's assets. Vives (2020) discusses common and cross-ownership structures that yield this objective. The parameter λ represents the weight each firm attributes to rival value. It is referred to as the *degree of internalization*, with $\lambda = 0$ representing purely self-interested behavior and $\lambda = 1$ representing joint value maximization. Backus et al. (2021) report average values for the degree of internalization up to .7 for U.S. firms. Estimates vary widely across both countries and industries, but the degree of internalization needn't vary much across firms. To motivate our assumption of symmetric λ, in U.S. pharmaceutical industry for example the fraction of total shares held as of August 2022 by the top three institutional shareholders (BlackRock, State Street and Vanguard) in the top three firms (Johnson & Johnson, Merck, and Pfizer) amounted to 19, 18, and 18% respectively.[1]

The market demand firms face is uncertain. At any time $t \geq 0$, inverse demand is

$$X(t)\left(1 - \eta Q(t)\right) \tag{2}$$

where $\eta > 0$, $Q(t)$ is industry capacity, and $X(t)$ is an exogenous shock. The exogenous

[1] Ownership structures are less likely to be symmetric in situations of cross-ownership, such as the minority share acquisitions which Heim et al. (2022) report.

shock evolves over time according to a geometric Brownian motion

$$dX(t) = \mu X(t)dt + \sigma X(t)d\omega(t) \tag{3}$$

where μ is the drift, $\sigma \geq 0$ the volatility, and $\omega(t)$ is a standard Wiener process.

The firms choose when and at what scale to enter the market. Market entry involves a single capacity investment. Capacity has a constant unit cost $\delta > 0$ and can be neither altered nor resold once it is installed. There are no production costs and firms are assumed to operate at capacity.[2]

Finally the discount rate r is constant with $r > \mu$ to focus on the case where the expected revenue stream is bounded.

3 Follower investment

Suppose one of the firms, the *leader*, invests a capacity Q_L and denote the current value of the demand state at that time by X. It is then up to the remaining firm, the *follower*, to choose when to invest and what capacity level Q_F to install when it enters. Letting T denote the follower's stopping time and $Q_F^*(T)$ its optimal capacity, the follower's perceived value is

$$\begin{aligned}\Omega_F(X) = \sup_{T\geq 0} E_X \Bigg[\lambda \int_0^T X(s)\left(1-\eta Q_L\right)Q_L e^{-rs}ds \\ + \int_T^\infty X(s)\left(1-\eta\left(Q_L+Q_F^*(T)\right)\right)\left(\lambda Q_L + Q_F^*(T)\right)e^{-rs}ds - \delta Q_F^*(T)e^{-rT}\Bigg)\Bigg]. \end{aligned} \tag{4}$$

[2] Ghemawat and Nalebuff (1985) explain how operating below capacity is technically inefficient in many real-world industries.

The conditional expectation in Eq. (4) has two parts. The first integral term is perceived discounted profit that accrues during the leader's monopoly phase, which lasts up until the stochastic time T at which the follower enters. The second set of terms is perceived net value upon entry, which consists of perceived discounted duopoly profit net of the follower's discounted investment cost.

The analysis of the follower's decision proceeds in two steps, first by characterizing the follower's capacity choice upon investment and then by determining its optimal timing.

To find the follower's optimal capacity, let $X' = X(T)$ denote the demand state at which the follower ultimately enters. Because $E_{X'}\left[\int_0^\infty X(s)e^{-rs}ds\right] = \frac{X'}{r-\mu}$, the perceived duopoly profit net of investment cost is

$$\frac{X'}{r-\mu}\left(1-\eta\left(Q_L+Q_F\right)\right)\left(\lambda Q_L+Q_F\right)-\delta Q_F \tag{5}$$

at time T. This expression is strictly concave in Q_F. Optimizing therefore gives a unique follower capacity upon investment which is a piecewise function of the state,

$$Q_F^*\left(X'\right) = \max\left\{0, \frac{1}{2\eta}\left(1-\eta\left(1+\lambda\right)Q_L-\frac{\delta\left(r-\mu\right)}{X'}\right)\right\}. \tag{6}$$

Eq. (6) indicates directly that internalization softens the follower's quantity reaction. Moreover, the leader can block the follower's entry permanently while obtaining a positive price if the degree of internalization is positive, by choosing a capacity $Q_L \in \left[\frac{1}{\eta(1+\lambda)}, \frac{1}{\eta}\right)$.

Substituting $Q_F^*\left(X'\right)$ back into the expected net present value expression (Eq. 5) gives the follower's payoff upon investment, which is

$$G_F\left(X'\right) = \frac{\left(1-\eta\left(1-\lambda\right)Q_L\right)^2}{4\eta}\frac{X'}{r-\mu}-\frac{\delta}{2\eta}\left(1-\eta\left(1+\lambda\right)Q_L\right)+\frac{\delta^2}{4\eta}\frac{r-\mu}{X'} \tag{7}$$

provided that $Q_L < \frac{1}{\eta(1+\lambda)}$ so the leader does not block entry.

With respect to the follower's entry timing, the follower holds a valuable real option if $Q_L < \frac{1}{\eta(1+\lambda)}$. This option involves both the terminal payoff in Eq. (7) and a perceived dividend flow $\lambda X(t)\,(1-\eta Q_L)\,Q_L$ stemming from the leader's monopoly position. A dynamic programming argument establishes that the follower's optimal policy is an investment threshold so that it invests once the demand state reaches

$$X_F^* = \begin{cases} \frac{\beta+1}{\beta-1}\frac{\delta(r-\mu)}{1-\eta(1+\lambda)Q_L}, & \text{if } Q_L < \frac{1}{\eta(1+\lambda)} \\ \infty, & \text{if } Q_L \geq \frac{1}{\eta(1+\lambda)} \end{cases} \tag{8}$$

where

$$\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{9}$$

is a constant that reflects discounting in a stochastic environment (see Appendix A.2). It is apparent from Eq. (8) that $\partial X_F^*/\partial\lambda > 0$, so greater internalization is associated with a higher investment threshold and consequently delayed entry. An increase in the degree of internalization thus raises the follower's perceived dividend relative to the perceived net duopoly payoff enough to delay investment. Together Eqs. (6) and (8) indicate that there is a monotonic effect of internalization on both the timing and size of the follower's investment, with λ effectively scaling up the leader's capacity by 100λ percent. Finally, if $X < X_F^*$ so the follower delays entry, its optimal capacity takes the value

$$Q_F^*\left(X_F^*\right) = \begin{cases} \frac{1-\eta(1+\lambda)Q_L}{(\beta+1)\eta}, & \text{if } Q_L < \frac{1}{\eta(1+\lambda)} \\ 0, & \text{if } Q_L \geq \frac{1}{\eta(1+\lambda)}. \end{cases} \tag{10}$$

The next proposition sets out the main results concerning follower investment.

Proposition 1. *The follower's investment threshold is X_F^* (Eq. 8), and its perceived* *value* $\Omega_F(X, Q_L)$ *is*

$$\begin{cases} \lambda\frac{(1-\eta Q_L)Q_L X}{r-\mu} + \frac{\delta(1-\eta(1+\lambda)Q_L)}{\left(\beta^2-1\right)\eta}\left(\frac{X}{X_F^*}\right)^\beta, & \textit{if } X < X_F^* \\ \frac{\left(1-\eta(1-\lambda)Q_L - \frac{\delta(r-\mu)}{X}\right)^2 X}{4\eta(r-\mu)}, & \textit{if } X \geq X_F^*, \end{cases} \tag{11}$$

if $Q_L < \frac{1}{\eta(1+\lambda)}$, *and*

$$\lambda\frac{(1-\eta Q_L)\, Q_L X}{r-\mu} \tag{12}$$

if $Q_L \geq \frac{1}{\eta(1+\lambda)}$.

The expressions in Proposition 1 apply once the leader has invested, and do not therefore account for perceived leader investment cost $\lambda\delta Q_L$ which is sunk at that time even if follower investment is immediate. If the leader does not block entry so that the follower's option is valuable, the follower value (Eq. 11) consists of two pieces. The first piece is its value if the demand state is low and it chooses to delay investment, so it obtains the sum of internalized leader value and its perceived option value. The second piece is the follower's perceived value if the demand state is high, so that it invests immediately. If the follower's real option is valueless, the follower's value (Eq. 12) consists only of internalized leader value.

4 Leader capacity choice

The leader's investment threshold cannot be lower than the net present value threshold for monopoly investment. We therefore can restrict attention to demand states $X >$

$\delta(r-\mu)$.[3] The leader's perceived value from investing at given X has the general form

$$\begin{aligned}\Omega_L(X) &= \max_{Q_L\geq 0} E_X\left[\int_0^T X(s)\left(1-\eta Q_L\right)Q_L e^{-rs}ds - \delta Q_L\right. \\ &\left.+\int_T^\infty X(s)\left(1-\eta\left(Q_L+Q_F^*\left(X(T)\right)\right)\right)\left(Q_L+\lambda Q_F^*\left(X(T)\right)\right)e^{-rs}ds - \lambda\delta Q_F^*\left(X(T)\right)e^{-rT}\right]\end{aligned} \tag{13}$$

where $T = \inf\{t\geq 0\,|X(t)\geq X_F^*\}$ is the follower's stopping time and $Q_F^*\left(X(T)\right)$ its capacity choice. Investment is assumed to be definitive, so the leader cannot reinvest at a later date if it chooses $Q_L = 0$.

In this section we study the leader's capacity choice upon investment. Because it affects the follower's entry timing and capacity, the leader's capacity choice is strategic. If the leader sets a small enough capacity, follower entry can be immediate (so $T = 0$) if the demand state is large enough. On the other hand if the leader sets a large enough capacity, the follower may never enter (so $T = \infty$). Intermediate capacity levels induce the follower to delay entry until a finite threshold $X_F^* > X$ is reached. Over the set of admissible capacities and demand states $\aleph = \left[0, \frac{1}{\eta}\right]\times\left(\delta\left(r-\mu\right),\infty\right)$, the expression of the conditional expectation in Eq. (13) depends on which of these three alternatives applies. We partition $\aleph$ according to follower entry behavior to obtain specific forms that allow us to study the leader's capacity choice problem.

[3] The value of monopoly investment in demand state X is

$$\begin{aligned}&\max_{Q\geq 0} E_X\left[\int_0^\infty X(s)\left(1-\eta Q\right)Q e^{-rs}ds - \delta Q\right] \\ =\ &\max_{Q\geq 0}\left[\frac{X}{r-\mu}\left(1-\frac{\delta\left(r-\mu\right)}{X}-\eta Q\right)Q\right]\end{aligned}$$

so monopoly capacity investment is positive only if $X > \delta\left(r-\mu\right)$.

First, for all $X \geq \widehat{X}_L$ where

$$\widehat{X}_L = \min_{Q_L \in \left[0, \frac{1}{\eta}\right]} X_F^* = \frac{\beta + 1}{\beta - 1} \delta \left(r - \mu\right) \tag{14}$$

bounds the set of possible follower entry thresholds from below, let

$$\widehat{Q}_L \left(X\right) = \frac{1}{\eta \left(1 + \lambda\right)} \left(1 - \frac{\beta + 1}{\beta - 1} \frac{\delta \left(r - \mu\right)}{X}\right) \tag{15}$$

denote the leader capacity at which the follower's threshold takes the value $X_F^* = X$. In (Q_L, X) space, the locus $\widehat{Q}_L(X)$ discriminates between those leader capacity levels at which follower entry is immediate and those at which it is delayed. Then, define

$$\begin{aligned} \aleph^{\mathrm{a}} &= \left\{ (Q_L, X) \in \aleph \middle| \, X \geq \widehat{X}_L \text{ and } Q_L \in \left[0, \widehat{Q}_L(X)\right] \right\}, \\ \aleph^{\mathrm{b}} &= \left\{ (Q_L, X) \in \aleph \middle| \, Q_L \geq \frac{1}{\eta (1 + \lambda)} \right\}, \text{ and} \\ \aleph^{\mathrm{d}} &= \aleph \setminus \left(\aleph^{\mathrm{a}} \cup \aleph^{\mathrm{b}} \right). \end{aligned} \tag{16}$$

If $(Q_L, X) \in \aleph^{\mathrm{a}}$ there is immediate duopoly, which can be interpreted as accommodation by the leader ($T = 0$). If $(Q_L, X) \in \aleph^{\mathrm{b}}$ the leader has a permanent monopoly and follower entry is blocked ($T = \infty$). Finally if $(Q_L, X) \in \aleph^{\mathrm{d}}$ there is delayed duopoly ($X < X_F^* < \infty$), which can be construed as a dynamic version of strategic deterrence. The forms that the conditional expectation term in Eq. (13) takes over $\aleph^{\mathrm{a}}$, $\aleph^{\mathrm{b}}$, and $\aleph^{\mathrm{d}}$ and the behavior with respect to Q_L over these regions are as follows.

$\aleph^{\mathrm{a}}$ **(immediate duopoly)**:

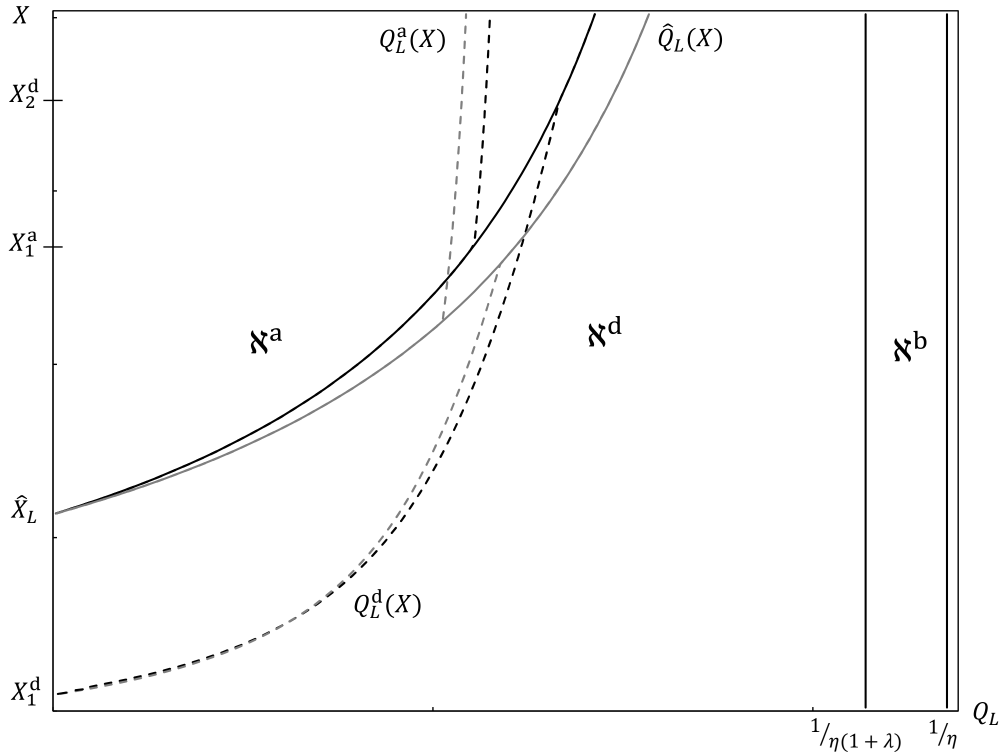

Figure 1: Leader capacity choice regions in (Q, X) space for $r = .1$, $\mu = .06$, $\sigma = .1$, $\delta = .1$, $\eta = .05$ and $\lambda = 0$ (gray) or .1 (black). Immediate duopoly is only possible at demand states above $\widehat{X}_L$. With internalization, the accommodation region $\aleph^{\mathrm{a}}$ shrinks and the delayed duopoly region $\aleph^{\mathrm{d}}$ expands. The dashed curves $Q_L^{\mathrm{a}}(X)$ and $Q_L^{\mathrm{d}}(X)$ respectively plot local capacity choice maxima with immediate or delayed duopoly.

In this region, $T = 0$ and the conditional expectation term has the form

$$E_X \left[\int_0^\infty X(s) \left(1 - \eta \left(Q_L + Q_F^* \left(X\right)\right)\right) \left(Q_L + \lambda Q_F^* \left(X\right)\right) e^{-rs} ds - \delta \left(Q_L + \lambda Q_F^* \left(X\right)\right) \right]. \tag{17}$$

This expectation is over the leader's perceived perpetual duopoly profit net of perceived investment cost. Although both firms effectively enter at the same moment, the follower's entry decision occurs "immediately after" the leader's. Because it observes the leader's investment and this investment is irreversible, the follower reacts to the leader's capacity through $Q_F^*(X)$. Evaluating the expectation gives

$$\frac{X\left(1-\eta\left(1-\lambda\right)Q_L-\frac{\delta(r-\mu)}{X}\right)\left(\lambda+\eta\left(2+\lambda\right)\left(1-\lambda\right)Q_L-\lambda\frac{\delta(r-\mu)}{X}\right)}{4\eta\left(r-\mu\right)}. \tag{18}$$

Viewed as a function of Q_L, Eq. (18) is concave with interior maximum

$$Q_L^{\mathrm{a}}(X)=\frac{1-\frac{\delta(r-\mu)}{X}}{\eta\left(2+\lambda\right)\left(1-\lambda\right)}. \tag{19}$$

For $\lambda<\sqrt{2}-1$, or for $\lambda>\sqrt{2}-1$ and $\beta>\frac{3-\lambda^2}{\left(\lambda+1-\sqrt{2}\right)\left(\lambda+1+\sqrt{2}\right)}$, there exists a unique demand state $X_1^{\mathrm{a}}>\widehat{X}_L$ at which $Q_L^{\mathrm{a}}(X)$ intersects $\widehat{Q}_L(X)$ from below in (Q,X) space. For demand states above X_1^{a}, the maximum is interior, at $Q_L^{\mathrm{a}}(X)$. Otherwise (for $\lambda=\sqrt{2}-1$ or $\lambda>\sqrt{2}-1$ and $\beta\leq\frac{3-\lambda^2}{\left(\lambda+1-\sqrt{2}\right)\left(\lambda+1+\sqrt{2}\right)}$), Eq. (18) is increasing and reaches its maximum on the boundary, at $\widehat{Q}_L(X)$. Solving the condition $Q_L^{\mathrm{a}}(X)=\widehat{Q}_L(X)$ gives

$$X_1^{\mathrm{a}}=\frac{\beta\left(2-(1+\lambda)^2\right)+3-\lambda^2}{\left(\beta-1\right)\left(2-(1+\lambda)^2\right)}\delta\left(r-\mu\right). \tag{20}$$

As λ increases, the boundary $\widehat{Q}_L(X)$ moves to the left. Greater internalization therefore shrinks the accommodation region, whereas the interior maximum $Q_L^{\mathrm{a}}(X)$ shifts rightward.

$\aleph^{\mathrm{d}}$ **(delayed duopoly):**

In this region, $T=\inf\{t>0\,|\,X(t)\geq X_F^*\}$ with a finite follower threshold $X_F^*>X$ and the conditional expectation term has the form

$$
\begin{aligned}
&E_X \left[\int_0^T X(s)(1-\eta Q_L)Q_L e^{-rs} ds - \delta Q_L \right. \\
&\left. + \int_T^\infty X(s)\left(1-\eta\left(Q_L + Q_F^*\left(X_F^*\right)\right)\right)\left(Q_L + \lambda Q_F^*\left(X_F^*\right)\right) e^{-rs} ds - \lambda \delta Q_F^*\left(X_F^*\right) e^{-rT} \right]
\end{aligned}
\tag{21}
$$

where X_F^* is the follower's threshold reaction function (Eq. 8 above). Inside this expression, the first two terms are the leader's discounted monopoly profit net of investment cost, and the last two terms are perceived discounted duopoly profit and internalized follower investment cost. Evaluating the expectation yields

$$
\frac{\left(1-\eta Q_L\right) Q_L X}{r-\mu} - \delta Q_L + \left(\frac{X}{X_F^*\left(Q_L\right)}\right)^\beta \frac{\delta\left(\lambda - \eta\left(1+\lambda\right)\left(\beta+1-\lambda\beta\right) Q_L\right)}{\eta\left(\beta^2-1\right)}. \tag{22}
$$

To characterize the behavior of Eq. (22) as leader capacity varies, differentiate with respect to Q_L to get the first-order condition at an interior optimum, which we denote by Q_L^{d},

$$
\frac{\left(1-2\eta Q_L^{\mathrm{d}}\right) X}{r-\mu} - \delta - \left(\frac{X}{X_F^*\left(Q_L^{\mathrm{d}}\right)}\right)^\beta \frac{\delta\left(1+\lambda\right)\left(1-\eta\left(1+\lambda\right)\left(\beta+1-\lambda\beta\right) Q_L^{\mathrm{d}}\right)}{\left(\beta-1\right)\left(1-\eta\left(1+\lambda\right) Q_L^{\mathrm{d}}\right)} = 0. \tag{23}
$$

The second-order condition is satisfied for β or λ small enough, and notably for $\lambda < \sqrt{\frac{\beta+1}{\beta}} - 1$. The range of demand states for which an interior optimum exists is determined as follows (see also Appendix A.3). First, setting $Q_L^{\mathrm{d}} = 0$ in Eq. (23) gives the lower bound of the demand states at which an interior optimum exists, X_1^{d}, as the lower root of

$$
\frac{\left(1+\lambda\right)\left(\beta-1\right)^{\beta-1}}{\left(\beta+1\right)^\beta} \left(\frac{X_1^{\mathrm{d}}}{\delta\left(r-\mu\right)}\right)^\beta - \frac{X_1^{\mathrm{d}}}{\delta\left(r-\mu\right)} + 1 = 0. \tag{24}
$$

The left-hand side of Eq. (24) is convex in the demand state and negative at $\widehat{X}_L$, so the lower root satisfies $X_1^{\mathrm{d}} < \widehat{X}_L$, and X_1^{d} increases unambiguously as λ increases. If $\lambda \notin \left[\sqrt{\frac{\beta+1}{\beta}} - 1, \frac{1}{\beta}\right]$, there exists a finite demand state at which setting $Q_L^{\mathrm{d}} = \widehat{Q}_L(X)$ solves the first-order condition (Eq. 23),

$$X_2^{\mathrm{d}} = \frac{(\beta+1)(2+\lambda)(1-\beta\lambda)}{(\beta-1)\left(1-2\beta\lambda-\beta\lambda^2\right)}\delta(r-\mu), \tag{25}$$

which represents the upper bound of the demand states at which an interior optimum exists. Otherwise if $\lambda \in \left[\sqrt{\frac{\beta+1}{\beta}} - 1, \frac{1}{\beta}\right]$ there is no such upper bound.

$\aleph^{\mathrm{b}}$ (permanent monopoly):

In this region $T = \infty$ and the conditional expectation term has the form

$$E_X\left[\int_0^\infty X(s)(1-\eta Q_L)Q_L e^{-rs}ds - \delta Q_L\right]. \tag{26}$$

Inside this expression is the perpetual monopoly profit net of investment cost. Evaluating the expectation gives

$$\left(\frac{X}{r-\mu}(1-\eta Q_L) - \delta\right)Q_L. \tag{27}$$

Because $\lambda \leq 1$, capacities in the $\aleph^{\mathrm{b}}$ region lie at or above the monopoly level. The leader's payoff is therefore decreasing in capacity, and hence maximized on the left boundary of this region by setting $Q_L = \frac{1}{\eta(1+\lambda)}$, for which the leader obtains

$$\Omega_L\left(\frac{1}{\eta(1+\lambda)}, X\right) = \frac{\lambda X}{\eta(r-\mu)(1+\lambda)^2} - \frac{\delta}{\eta(1+\lambda)}. \tag{28}$$

To summarize, the payoff $\Omega_L(Q_L, X)$ which the leader obtains from choosing capacity Q_L upon investment is given by the following proposition.

Proposition 2. *At the capacity choice stage,*

$\Omega_L(Q_L, X) =$

$$\begin{cases} \frac{(1-\eta)Q_L X}{r-\mu} - \delta Q_L + \left(\frac{X}{X_F^*}\right)^\beta \frac{\delta(\lambda-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L)}{\eta\left(\beta^2-1\right)}, & \textit{if } 0 \leq Q_L < \frac{1}{\eta(1+\lambda)} \\ \left(\frac{X}{r-\mu}(1-\eta Q_L) - \delta\right) Q_L, & \textit{if } \frac{1}{\eta(1+\lambda)} \leq Q_L \leq \frac{1}{\eta} \end{cases} \tag{29}$$

if $X < \widehat{X}_L$,

and

$$\begin{cases} \frac{X\left(1-\eta(1-\lambda)Q_L - \frac{\delta(r-\mu)}{X}\right)\left(\lambda+\eta(2+\lambda)(1-\lambda)Q_L - \lambda\frac{\delta(r-\mu)}{X}\right)}{4\eta(r-\mu)}, & \textit{if } 0 \leq Q_L \leq \widehat{Q}_L(X) \\ \frac{(1-\eta Q_L)Q_L X}{r-\mu} - \delta Q_L + \left(\frac{X}{X_F^*}\right)^\beta \frac{\delta(\lambda-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L)}{\eta\left(\beta^2-1\right)}, & \textit{if } \widehat{Q}_L(X) < Q_L < \frac{1}{\eta(1+\lambda)} \\ \left(\frac{X}{r-\mu}(1-\eta Q_L) - \delta\right) Q_L, & \textit{if } \frac{1}{\eta(1+\lambda)} \leq Q_L \leq \frac{1}{\eta} \end{cases} \tag{30}$$

if $X \geq \widehat{X}_L$.

In Proposition 2, the first part (Eq. 29) states that at low demand states the leader's capacity choice problem involves two pieces corresponding to the delay and blockade regions. The second part (Eq. 30) states that the payoff at high demand states consists of these two pieces along with an additional low capacity range where accommodation occurs.

As discussed in the preceding paragraphs, the conditions under which the leader's payoff admits a local maximum and under which such a maximum is interior over each piece depend on the level of the demand state. Moreover because the optimal capacity under delay Q_L^{d} is defined only implicitly, there is no analytic solution to the capacity choice problem $\max_{Q_L \in [0,1/\eta]} \Omega_L(Q_L, X)$. In some cases however, the set of candidate solutions can be significantly narrowed.

To better visualize the leader's capacity choice problem, Figure 1 depicts $\aleph$ for a low degree of internalization ($\lambda = .1$, in black) and without internalization ($\lambda = 0$, in grey).

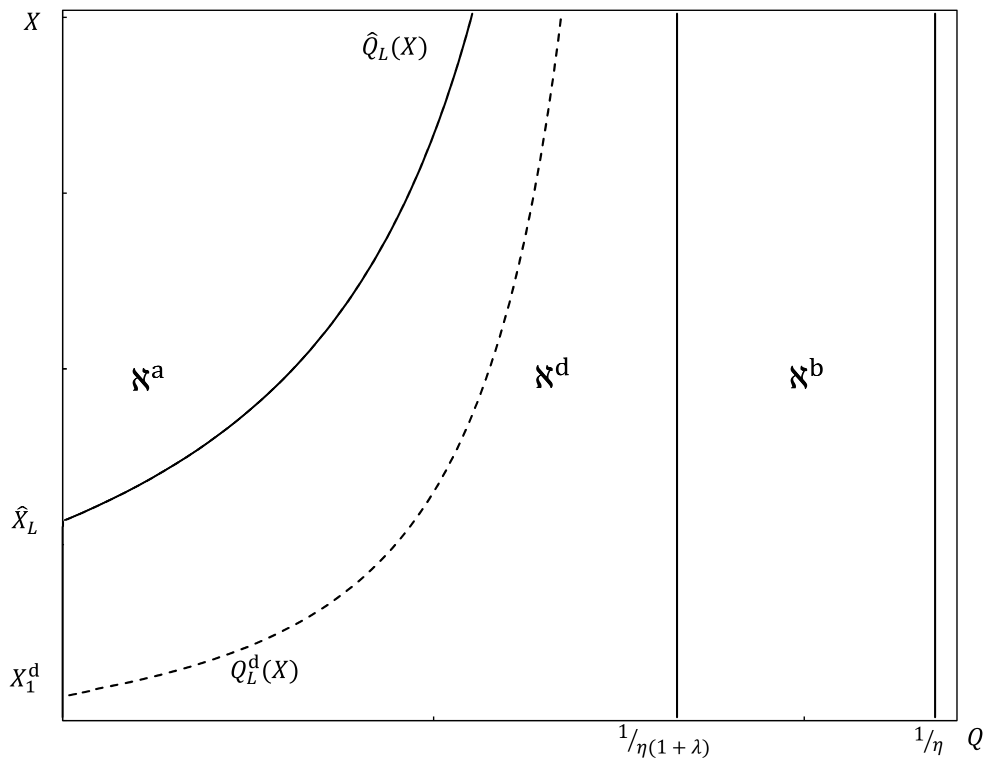

Figure 2: Leader capacity choice regions in (Q, X) space for $r = .1$, $\mu = .06$, $\sigma = .1$, $\delta = .1$, $\eta = .05$ and $\lambda = .42$. With these parameter values there is no local maximum with immediate duopoly whereas a local maximum with delayed duopoly $Q_L^{\mathrm{d}}(X)$ exists for all demand states above X_1^{d}, indicated by the dashed curve.

The configuration is similar for all levels of internalization $\lambda < \sqrt{\frac{\beta+1}{\beta}} - 1$ at which X_1^{a} and X_2^{d} are finite, with $X_1^{\mathrm{a}} < X_2^{\mathrm{d}}$. The three regions $\aleph^{\mathrm{a}}$, $\aleph^{\mathrm{b}}$, and $\aleph^{\mathrm{d}}$ are delimited by solid curves. On the left, $\aleph^{\mathrm{a}}$ and $\aleph^{\mathrm{d}}$ are similar to the standard case without internalization whereas $\aleph^{\mathrm{b}}$ is specific to our framework. The local maxima Q_L^{a} and Q_L^{d}, where they exist, are plotted with dashed curves. Over the range of demand states $\left(X_1^{\mathrm{a}}, X_2^{\mathrm{d}}\right)$ therefore, the leader's capacity choice problem has multiple local optima.

With higher levels of internalization, the leader's strategic behavior differs markedly from the situation without internalization. For example if $\lambda > \sqrt{2}-1$ and $\beta \leq \frac{1}{\lambda}$, both X_1^{a}

and X_2^{d} are undefined which implies that the relevant strategies for the leader induce either delayed duopoly or permanent monopoly, but never accommodation. Figure 2 illustrates this possibility where the blockade region is significantly expanded, accommodation is possible in principle for any demand state above $\widehat{X}_L$ but takes the form of a corner solution at $\widehat{Q}_L$ (which is never optimal), so the leader invariably chooses strategic delay for any demand state above X_1^{d}.

At low levels of internalization, the leader's capacity choice problem resembles the case without internalization insofar as the regions $\aleph^{\mathrm{a}}$ and $\aleph^{\mathrm{d}}$ and the local maxima Q_L^{a} and Q_L^{d} shift on the margin. Blockading the follower turns out not to be an optimal strategy, because it is more efficient for the leader to delay strategically than to blockade at low demand states whereas accommodation is preferable thereafter. The following proposition describes these ideas.

Proposition 3. *For $\lambda < \sqrt{\frac{\beta+1}{\beta}} - 1$, the leader's payoff from the capacity choice stage is*

$$\Omega_L(X) = \begin{cases} \frac{\lambda(\beta-1)^{\beta-1}X^\beta}{(\beta+1)^{\beta+1}\eta\delta^{\beta-1}(r-\mu)^\beta}, & \text{if } \delta\,(r-\mu) \le X \le X_1^{\mathrm{d}} \\ \frac{\left(1-\eta Q_L^{\mathrm{d}}\right)Q_L^{\mathrm{d}}X}{r-\mu} - \delta Q_L^{\mathrm{d}} + \left(\frac{X}{X_F^*}\right)^\beta \frac{\delta\left(\lambda-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L^{\mathrm{d}}\right)}{\eta\left(\beta^2-1\right)}, & \text{if } X_1^{\mathrm{d}} < X < X_1^{\mathrm{a}} \\ \max\left\{\frac{(1+\lambda)^2X}{4\eta(2+\lambda)(r-\mu)}\left(1-\frac{\delta(r-\mu)}{X}\right)^2, \frac{\left(1-\eta Q_L^{\mathrm{d}}\right)Q_L^{\mathrm{d}}X}{r-\mu} - \delta Q_L^{\mathrm{d}} \right. & \\ \qquad \left. + \left(\frac{X}{X_F^*}\right)^\beta \frac{\delta\left(\lambda-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L^{\mathrm{d}}\right)}{\eta\left(\beta^2-1\right)}\right\}, & \text{if } X_1^{\mathrm{a}} \le X < X_2^{\mathrm{d}} \\ \frac{(1+\lambda)^2X}{4\eta(2+\lambda)(r-\mu)}\left(1-\frac{\delta(r-\mu)}{X}\right)^2, & \text{if } X \ge X_2^{\mathrm{d}}. \end{cases} \tag{31}$$

Proof. See Appendix A.4.

The main consequence of this proposition is that with lower degrees of internalization the solution of the leader's capacity choice problem resembles the situation without internalization. That is to say, despite the possibility of a blockade strategy, the only optimal capacities are either those which induce delay (0 or Q_L^{d}) or an interior accommodation solution (Q_L^{a}). In addition the relevant thresholds satisfy $X_1^{\mathrm{d}} < X_1^{\mathrm{a}} < X_2^{\mathrm{d}}$ just as they do in the absence of internalization. The suboptimality of blockade is all the more striking at high demand states ($X \geq X_2^{\mathrm{d}}$) where the outcomes of the leader's capacity decision converge towards those of the static model so that the leader's strategies might be expected to include blockade as in the standard entry deterrence model so the outcome of its capacity choice might be expected to be accommodation or blockade as in the standard entry deterrence model of Dixit (1980).

In the absence of internalization, Huisman and Kort (2015) observe that the accommodation and delay payoffs cross once in $\left(X_1^{\mathrm{a}}, X_2^{\mathrm{d}}\right)$, so that the leader chooses delayed duopoly at lower demand states and accommodation at higher demand states, with an upper bound of the range over which the leader opts for deterrence that we denote by $\overline{X}_L^{\mathrm{d}}$. Intuitively, because internalization softens both the follower's timing and capacity choice, we expect internalization to favor deterrence (where both dimensions of the follower's reaction matter) relative to accommodation (where only quantity choice matters) so that $\overline{X}_L^{\mathrm{d}}$ increases. The numerical analysis we conduct in Section 6 also bears out these ideas.

5 Equilibrium investment

To characterize equilibrium investment in an early stage of the industry, suppose that the roles of each firm (leader or follower) are determined noncooperatively (implying $\lambda < 1$)[4] and that the initial demand state is low enough that no firm invests immediately.[5] At

[4] If $\lambda = 1$ joint profit is maximized by having a single firm invest as a monopoly, i.e. with capacity $\frac{1}{\eta(\beta+1)}$ at the demand state threshold $\frac{\beta+1}{\beta-1}\delta(r-\mu)$.

[5] The demand state is sufficiently low for firms to wait if $X \leq X_P$, where X_P is defined further below in the section. In addition we assume that the delayed duopoly capacity Q_L^{d} is well-defined over $\left(X_1^{\mathrm{d}}, X_2^{\mathrm{d}}\right)$,

any demand state X at which no investment has yet occurred, the firms have the choice to invest or to wait. The instantaneous payoff from investing as a leader is the payoff $\Omega_L(X)$ described in the preceding section (e.g. Eq. 31 if the degree of internalization is sufficiently small). If the rival invests at X on the other hand, the instantaneous payoff for the remaining firm is the follower payoff $\Omega_F(X, Q_L^*)$ in Eq. (11) net of the internalized leader investment cost $\lambda\delta Q_L^*$, where Q_L^* denotes the leader's optimal capacity at X. The incentive of each firm to preempt its rival is therefore given by the payoff difference

$$f(X) = \Omega_L(X) - \left(\Omega_F(X, Q_L^*) - \lambda\delta Q_L^*\right). \tag{32}$$

The set of demand states over which firms prefer to lead rather than follow is called the preemption range. In equilibrium, the first investment in the industry takes place at the lower bound of this range. We denote this lower bound by X_P. Intuitively, since X_P is the smallest demand state at which firms prefer to lead rather than follow and as the payoff to following is non-negative, if one firm were to set a higher investment threshold $X' > X_P$ its rival would have an incentive to enter before it at a lower threshold in (X_P, X'). If the initial state is low enough therefore ($X < X_P$), in equilibrium one firm must invest as a leader at X_P.[6] The preemption threshold does not have an explicit expression but the following proposition identifies a range for it and establishes that firms invest sequentially in equilibrium.

Proposition 4. *For initial states $X \leq X_1^{\mathrm{d}}$, in a preemption equilibrium the leader invests at the demand state threshold $X_P = \inf\{X > 0,\ s.t.\ f(X) > 0\}$ satisfying $X_P \in \left(X_1^{\mathrm{d}}, \min\left\{X_1^{\mathrm{a}}, X_2^{\mathrm{d}}\right\}\right)$ and chooses capacity $Q_L^{\mathrm{d}}(X_P)$. The follower invests at the demand*

e.g. because the second-order condition (Eq. 54) is satisfied.

[6]This is a simplified description of preemption. See Thijssen et al. (2012) for a formalization of this game specifying an appropriate strategy space and outcomes if both firms invest at the same threshold.

state threshold X_F^* and chooses capacity $Q_F^*\left(X_F^*\right)$.

Proof. See Appendix A.5.

Propositions 4 together with Proposition 3 implies that for small enough internalization levels, the first equilibrium investment occurs at a threshold in $\left(X_1^{\mathrm{d}}, X_1^{\mathrm{a}}\right)$ where the leader's optimum capacity is $Q_L^* = Q_L^{\mathrm{d}}$ so the follower's investment is delayed. In this range the preemption incentive takes the form

$$
\begin{aligned}
f(X) = (1-\lambda)\Bigg(& \frac{\left(1-\eta Q_L^{\mathrm{d}}\right) Q_L^{\mathrm{d}} X}{r-\mu} - \delta Q_L^{\mathrm{d}} - \left(\frac{X}{X_F^*}\right)^{\beta}\left(\frac{\left(1-\eta Q_L^{\mathrm{d}}\right) Q_L^{\mathrm{d}} X_F^*}{r-\mu} - \delta Q_L^{\mathrm{d}}\right) \\
& + \left(\frac{X}{X_F^*}\right)^{\beta}\left(\frac{\left(1-\eta\left(Q_L^{\mathrm{d}}+Q_F^*\left(X_F^*\right)\right)\right) X_F^*}{r-\mu} - \delta\right)\left(Q_L^{\mathrm{d}} - Q_F^*\left(X_F^*\right)\right)\Bigg).
\end{aligned}
\tag{33}
$$

Up to scaling by $1-\lambda$, Eq. (33) breaks the preemption incentive down into two parts. The first part is the rent that the leader obtains from the industry's monopoly phase by entering ahead of the follower with capacity Q_L^{d}. The second part consists of the terms in the second line, which represent the leader's profit relative to the follower's during the industry's duopoly phase. This relative profit is positive if the leader has a larger capacity than the follower and negative if the reverse is true.

To see how internalization affects the preemption incentive, recall from Section 3 that internalization softens the follower's investment timing and quantity reactions. Because the follower acts less aggressively, a leader benefits from internalization, both through its lengthier monopoly phase and through the higher duopoly share it gets once the follower does enter. These positive effects of internalization are offset, from the leader's perspective, by lower perceived follower value. Similarly, from the follower's perspective the decrease in its own value with internalization is offset by higher perceived leader value. Overall we would expect own value effects to dominate at least at low levels of internalization, so a small increase in internalization should raise the preemption incentive and result in earlier initial investment. The counterpart to the anticompetitive effect of overlapping ownership on follower behavior is thus that firms act more competitively ex-ante. The implicit expressions for Q_L^{d} and X_P prohibit showing this in full generality, but in the next proposition we establish that internalization does have such procompetitive effect on the preemption equilibrium at low levels of internalization.

Proposition 5. *For small enough λ, the preemption threshold and leader capacity decrease with internalization ($dX_P/d\lambda, dQ_L^{\mathrm{d}}\left(X_P\right)/d\lambda < 0$).*

Proof. See Appendix A.6.

The negative effect of internalization on the preemption threshold is similar to a corresponding result in the fixed investment size case where a weaker follower timing reaction due to internalization accelerates preemptive investment (Zormpas and Ruble 2021). The negative capacity effect in Proposition 5 shows that the procompetitive effect on timing must be nuanced if capacities are endogenized. At a high enough demand state the leader has an incentive to raise capacity because the follower reacts less aggressively, but in the preemption equilibrium this strategic capacity effect is dominated by the decrease in equilibrium threshold which drives the leader to lower its capacity overall.

In the next section, we verify in an example that moderate internalization leads to similar outcomes, with both earlier investment and lower leader capacity. The overall effect of internalization on welfare is generally involved because of the contrasting effects on leader timing and capacity as well as on the follower's behavior. In the numerical analyses we conduct, we find the procompetitive effect of moderate internalization on leader timing is offset by these other effects, so consumer surplus ultimately decreases.

6 Numerical analysis

As the leader's optimal capacity and the preemption threshold are defined only implicitly, we use numerical methods in this section to further examine the consequences of internalization. We use the parameter values $r = .1$, $\mu = .06$, $\sigma = .1$, $\delta = .1$ and $\eta = .05$ as in Huisman and Kort (2015) and measure the effects of moderate internalization levels ($\lambda = .1$ or $\lambda = .2$). We conducted the same computations varying the values for the discount rate, drift, and volatility parameters and obtained similar results to those we report here. Besides corroborating the main insights of the preceding sections, e.g. an anticompetitive effect of internalization on follower behavior and a procompetitive effect on equilibrium investment, the numerical analysis also serves to highlight a novel procompetitive effect of internalization at demand states above the preemption threshold, whereby moderate internalization drives a leader to opt for strategic deterrence by choosing significantly larger capacity.

To visualize first how internalization affects equilibrium investment timing, Figure 3 plots the leader and follower payoffs as functions of the demand state. The leader payoff $\Omega_L(X)$ is the upper envelope of the payoffs under accommodation and delay, i.e. of the local maxima $\Omega_L\left(\min\left\{Q_L^{\mathrm{a}}(X), \widehat{Q}(X)\right\}, X\right)$ (dashed curve) and $\Omega_L(Q_L^{\mathrm{d}}(X), X)$ (dotted curve). With internalization, the leader perceives a positive payoff even below X_1^{d} because it accounts for the follower's positive option value. The follower's ex-ante payoff lies above the leader payoff initially, and crosses below it at the preemption threshold X_P. The figure indicates there is a single demand state $\overline{X}_L^{\mathrm{d}}$ at which the leader shifts from deterrence to accommodation. This shift creates an upward kink in the leader payoff and an upward jump in the follower payoff. The effect of internalization is gauged by comparing with the benchmark no-internalization case which is plotted in gray. With respect to the two

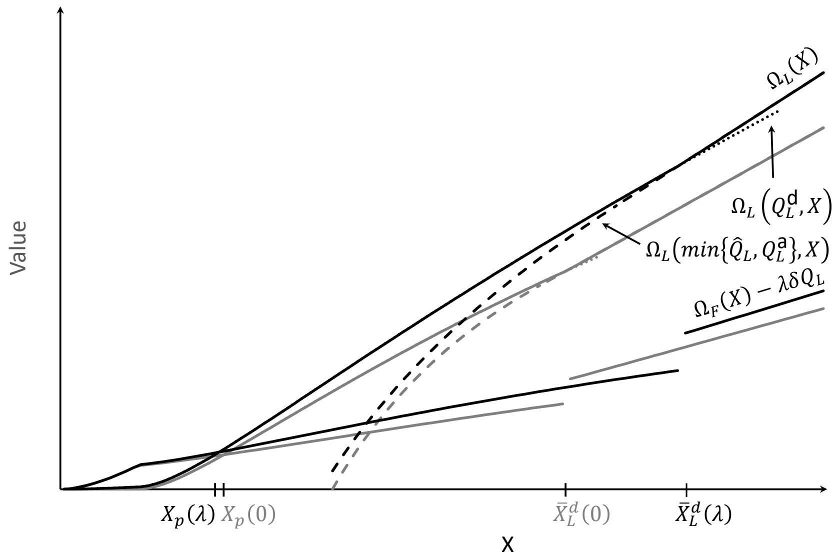

Figure 3: Leader and follower payoffs for $r = .1$, $\mu = .06$, $\sigma = .1$, $\delta = .1$, $\eta = .05$ and $\lambda = 0$ (gray) or $.1$ (black). At the demand state $\overline{X}_L^{\mathrm{d}}$ the leader shifts from deterrence to accommodation, resulting in a kink in the leader payoff and an upward jump in the follower payoff. The preemption equilibrium X_P lies at the intersection of the leader and follower payoffs. Greater internalization results in earlier equilibrium investment (lower X_P) and drives the leader to pursue deterrence over a broader range (higher $\overline{X}_L^{\mathrm{d}}$).

critical demand states, X_P decreases with internalization consistently with Proposition 5, whereas $\overline{X}_L^{\mathrm{d}}$ increases so the leader chooses to deter the follower over a broader range of demand states.

The effect of internalization on firm capacities is represented in Figure 4, which plots optimal leader and follower capacities against the demand state at which the leader invests. For either of the strategies that a leader can adopt (deterrence or accommodation), higher demand states result in higher leader capacity. At the demand state $\overline{X}_L^{\mathrm{d}}$ where the leader shifts from deterrence to accommodation however, its optimal capacity jumps downward. The follower's capacity is decreasing in the demand state if the leader opts to delay its entry, but increasing if both firms invest simultaneously. The effect of internalization on capacities is non-monotonic. Internalization decreases the leader's optimal capacity at low demand states though the effect is slight, and increases it at higher demand states. The follower's equilibrium capacity on the other hand increases with internalization at low demand states, albeit very slightly, and decreases at higher demand states.

For a given leader strategy, total capacity (hence static consumer surplus) is invariably lower with internalization. However, with the moderate internalization levels considered here, over the range of demand states where internalization shifts the leader's strategy from accommodation to deterrence, the procompetitive effect of internalization on leader capacity outweighs the anticompetitive effect on follower capacity and total capacity increases. Thus, if a leader is brought to invest at a moderately high demand state (e.g. because the initial value of $X(t)$ is sufficiently high in the preemption game), moderate internalization can shift its strategy towards deterrence so as to induce a sufficiently higher leader capacity that total capacity increases. Static welfare therefore increases during the industry's duopoly phase, but there is also a countervailing dynamic effect because the follower's entry is delayed until its optimal threshold X_F^* is reached.

To evaluate the effect of internalization on welfare, we assume that consumers have the same discount rate as firms and use a consumer surplus welfare standard, which is stricter than total surplus as internalization invariably raises firm values in equilibrium. The consumer surplus at the moment that the leader invests is

$$S(X) = E_X \left[\int_0^T \frac{1}{2} X(s) \eta Q_L^2 e^{-rs} ds + \int_T^\infty \frac{1}{2} X(s) \eta \left(Q_L + Q_F^* \left(X(T) \right) \right)^2 e^{-rs} ds \right]. \quad (34)$$

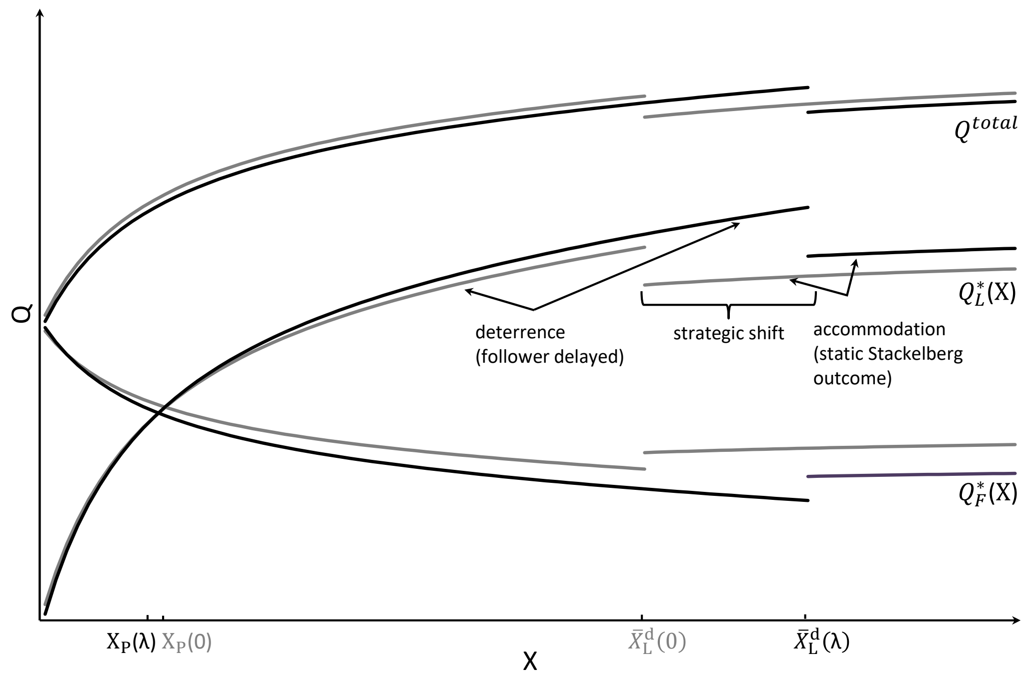

Figure 4: Optimal leader and follower capacities for $r = .1$, $\mu = .06$, $\sigma = .1$, $\delta = .1$, $\eta = .05$ and $\lambda = 0$ (gray) or $.1$ (black). For demand states between $\overline{X}_L^{\mathrm{d}}(0)$ and $\overline{X}_L^{\mathrm{d}}(\lambda)$ internalization shifts the leader's strategy from accommodation to deterrence, which results here in higher total output.

Inside the conditional expectation in Eq. (34), the first term is the discounted consumer surplus during the industry's monopoly phase and the second term is the discounted consumer surplus during the industry's duopoly phase, evaluated at an optimal leader capacity. If the demand state is sufficiently high for the leader to choose positive capacity and low enough for it to opt for delay, taking the expectation gives

$$S^{\mathrm{d}}(X) = \frac{\eta}{2}\frac{\left(Q_L^{\mathrm{d}}\right)^2 X}{r-\mu} + \frac{\eta}{2}\left(\frac{X}{X_F^*}\right)^{\beta}\frac{\left(\left(Q_L^{\mathrm{d}} + Q_F^*\left(X_F^*\right)\right)^2 - \left(Q_L^{\mathrm{d}}\right)^2\right)X_F^*}{r-\mu}. \tag{35}$$

At demand states which are high enough that the leader accommodates and follower entry is immediate, substituting values for $Q_F^*(X)$ and Q_L from Eqs (6) and (19) gives a consumer surplus expression

$$S^{\mathrm{a}}(X) = \frac{(3+\lambda)^2}{8\eta\,(2+\lambda)^2}\left(1-\frac{\delta\,(r-\mu)}{X}\right)^2\frac{X}{r-\mu}. \tag{36}$$

We first study the preemption equilibrium and resulting welfare. Table 2 reports the values which we obtain. Consistently with Proposition 5 in the preceding section, internalization has a procompetitive effect on equilibrium entry timing, but also results in lower leader capacity. The effect on the follower's capacity is negative whereas its threshold increases. Internalization therefore raises instantaneous consumer surplus because the monopoly phase starts earlier, but also has countervailing consequences on leader capacity and on the timing and surplus associated with the duopoly phase. To gauge the overall consumer surplus effect, we take three different degrees of internalization (0, .1, and .2) and determine the preemption equilibrium in each case. To compare consumer surplus values we evaluate these at a common demand state $X_P(.2) = .0100$, which is the smallest of the preemption equilibria. In the present case, the countervailing effects dominate here so the effect of moderate internalization levels on consumer surplus is negative.

Table 2: Preemption equilibrium

λ	$X_P(\lambda)$	Q_L^*	Q_F^*	Q^{Total}	X_F^*	$S(X)\vert_{X=X_P(.2)}$
0	.0105	5.53	5.59	11.12	0.0243	.5269
.1	.0102	5.38	5.44	10.82	0.0250	.4967
.2	.0100	5.22	5.31	10.52	0.0256	.4678

We focus next on the strategic shift effect. To this end we take initial demand states around the range where internalization alters the leader's strategy and suppose that one of the firms invests immediately as a leader in the initial state, either exogenously or as a result of competition with the follower. A shift in leader strategy has contrasting effects on overall consumer surplus, as it raises capacity which mitigates the harm induced by the monopoly phase (which would not be incurred if the leader accommodated), but also raises capacity during the industry's duopoly phase. To assess the balance of these effects, we take three different degrees of internalization (0, .1, and .2) and evaluate consumer surplus at the three demand states to the right of which the leader's strategy shifts ($\overline{X}_L^{\mathrm{d}}(0) = .0325$, $\overline{X}_L^{\mathrm{d}}(.1) = .0397$, and $\overline{X}_L^{\mathrm{d}}(.2) = .0559$) The resulting capacities and surplus values are reported in Table 3.

In the top part of the table which corresponds to a demand state just to the left of $\overline{X}_L^{\mathrm{d}}(0)$, the leader chooses deterrence for all the internalization levels, as evidenced by the fourth column (X_F^* is invariably larger than X). As described above, with the leader's strategy held constant, leader capacity increases with internalization whereas the follower's capacity decreases, and both total duopoly output and consumer surplus decrease with internalization.

The middle part of the table is the most similar to Table 1 in the introduction, though in the dynamic model the effect of internalization is necessarily more involved as it manifests itself through both capacities and follower timing. The leader chooses accommodation if $\lambda = 0$ and deterrence otherwise, so internalization produces a strategic shift here. As a result, the leader's capacity and total capacity increase sharply between $\lambda = 0$ and $\lambda = .1$, yielding in an increase in consumer surplus of roughly 5%. Going from $\lambda = .1$ to $\lambda = .2$, the increase in leader capacity is much smaller and total capacity decreases. Total capacity is still higher than without internalization, but the follower's investment is significantly delayed and consumer surplus decreases.

Table 3: Procompetitive strategic shift (dynamic)

	$X = .0325$				
λ	Q_L^*	Q_F^*	Q^{Total}	X_F^*/X	$S(X)$
0	9.75	3.96	13.71	1.06	3.76
.1	10.09	3.44	13.53	1.22	3.54
.2	10.29	2.96	13.24	1.41	3.30

	$X = .0397$				
λ	Q_L^*	Q_F^*	Q^{Total}	X_F^*/X	$S(X)$
0	8.99	4.50	13.49	1	4.51
.1	10.77	3.15	13.92	1.09	4.72
.2	10.96	2.65	13.60	1.29	4.37

	$X = .0559$				
λ	Q_L^*	Q_F^*	Q^{Total}	X_F^*/X	$S(X)$
0	9.28	4.64	13.93	1	6.78
.1	9.82	3.88	13.71	1	6.56
.2	12.04	2.14	14.19	1.13	6.89

In the bottom part of the table, the leader's strategy does not shift at $\lambda = .1$ so a low level of internalization only leads to reduced total capacity and consumer surplus. At $\lambda = .2$ however, the shift to deterrence does occur and the sharp increase in leader capacity that ensues results in slightly higher welfare than without internalization, again because the increase in the follower's threshold is not too large.

We take from these results that increased competition for the market due to internalization may not be beneficial to consumers if capacities are endogenous, but also that if competition occurs at a moderately high demand state some degree of common ownership or cross holdings can generate a procompetitive shift in the leader's strategy which is beneficial for consumers.

7 Conclusion

In this article, we study how common ownership or symmetric cross holdings affect strategic capacity decisions in an evolving market by driving managers to internalize effects on rival firms. Greater internalization predictably causes a follower to react less aggressively with respect to both its timing and capacity decisions. Once a leader has invested these effects may be construed as anticompetitive. But if firms have endogenous roles and must compete for industry leadership, then the follower's softer timing and capacity reactions raise the attractiveness of leadership, exerting a procompetitive effect on initial entry. Moreover, because softer timing and capacity reactions are particularly beneficial to the leader if the follower delays entry, internalization drives leaders to pursue deterrence over a broader range of demand states. We show through an example that internalization can thus generate a procompetitive shift in leader strategy which moreover can benefit consumers, so that the same effect discussed in the two-stage model in the introduction also arises in a more comprehensive dynamic model. In addition, because they do not hinge on the presence of spillovers or high R&D intensity, the procompetitive effects of overlapping ownership which we identify are liable to apply in a broad range of industries.

Our analysis relies on several assumptions which could be relaxed in future work. To begin with, the assumption of symmetric ownership structures may closely reflect common ownership in certain industries but not in others. In the case of unilateral minority share acquisitions for example, cross holdings are naturally asymmetric. A closer representation of these situations would therefore account for asymmetric ownership and result in an asymmetric preemption game, whose equilibrium outcome generalizes the one which we describe here. In addition, we have restricted our attention to new markets where neither firm initially operates, but further effects of overlapping ownership could be expected to arise in markets where firms have preexisting capacities or the ability to make multiple capacity additions.

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A Appendix

A.1 Internalization in the Stackelberg-Spence-Dixit framework

For simplicity, let inverse demand be $P = 1 - Q$ and suppose that production cost is zero. Firms have a symmetric ownership structure resulting in a degree of internalization $\lambda < 1$. The firms choose capacities sequentially.

With these assumptions, the follower's perceived profit function is

$$\begin{aligned} \omega_F\left(Q_F, Q_L\right) &= \pi_F\left(Q_F, Q_L\right) + \lambda \pi_L\left(Q_F, Q_L\right) \\ &= \left(1 - Q_L - Q_F\right)\left(Q_F + \lambda Q_L\right). \end{aligned} \tag{37}$$

The follower's optimal capacity is

$$Q_F^*(Q_L) = \max\left\{0, \frac{1 - (1+\lambda)\, Q_L}{2}\right\}, \tag{38}$$

so internalization softens the follower's quantity reaction (as occurs in the dynamic model).

Its perceived profit is

$$\omega_F\left(Q_F^*(Q_L), Q_L\right) = \begin{cases} \frac{(1-(1-\lambda)Q_L)^2}{4}, & \text{if } Q_L < \frac{1}{1+\lambda} \\ \lambda\left(1-Q_L\right)Q_L, & \text{if } Q_L \geq \frac{1}{1+\lambda}. \end{cases} \tag{39}$$

Given the follower's reaction $Q_F^*(Q_L)$, the leader's perceived profit is therefore

$$\begin{aligned} \omega_L\left(Q_F, Q_L\right) &= \pi_L\left(Q_F, Q_L\right) + \lambda\pi_F\left(Q_F, Q_L\right) \\ &= \left(1-\left(Q_L + Q_F^*(Q_L)\right)\right)\left(Q_L + \lambda Q_F^*(Q_L)\right). \end{aligned} \tag{40}$$

It is straightforward to check $\frac{\partial\omega_L}{\partial Q_F}\left(Q_F, Q_L\right) > 0$, so internalization raises the incentive to lead by weakening the follower's reaction. Evaluating the preceding expression gives

$$\omega_L\left(Q_F, Q_L\right) = \begin{cases} \frac{(1-(1-\lambda)Q_L)(\lambda+(2+\lambda)(1-\lambda)Q_L)}{4}, & \text{if } Q_L < \frac{1}{1+\lambda} \\ \left(1-Q_L\right)Q_L, & \text{if } Q_L \geq \frac{1}{1+\lambda}. \end{cases}$$

Because $\frac{1}{1+\lambda} > \frac{1}{2}$, π_L is decreasing over the second piece, and for large enough values of λ the maximum is reached at $Q_L = \frac{1}{1+\lambda}$. The optimal leader capacity is therefore

$$Q_L^{\mathrm{a}} = \begin{cases} \frac{1}{(2+\lambda)(1-\lambda)}, & \text{if } \lambda < \sqrt{2}-1 \\ \frac{1}{1+\lambda}, & \text{if } \lambda \geq \sqrt{2}-1. \end{cases} \tag{41}$$

This results in a leader profit

$$\omega_L\left(Q_F^*\left(Q_L^{\mathrm{a}}\right), Q_L^{\mathrm{a}}\right) = \begin{cases} \frac{(1+\lambda)^2}{4(2+\lambda)}, & \text{if } \lambda < \sqrt{2}-1 \\ \frac{\lambda}{(1+\lambda)^2}, & \text{if } \lambda \geq \sqrt{2}-1. \end{cases} \tag{42}$$

Thus, if $\lambda \geq \sqrt{2}-1$, $Q_F^*(Q_L^*) = 0$ and follower entry is deterred entirely by internalization.

Hereafter, we focus on the case $\lambda < \sqrt{2} - 1$ and introduce a fixed cost of entry $f > 0$.

The follower enters if

$$\omega_F \left(Q_F^*(Q_L), Q_L\right) - f > \omega_F \left(0, Q_L\right), \tag{43}$$

which holds if

$$Q_L < Q_L^{\mathrm{d}} = \frac{1 - 2\sqrt{f}}{1 + \lambda}. \tag{44}$$

The follower's entry is naturally blockaded if $Q_L^{\mathrm{d}} \leq \frac{1}{2}$, i.e. $\lambda > 1 - 4\sqrt{f}$. Otherwise, the leader prefers deterrence over accommodation if

$$\omega_L \left(0, Q_L^{\mathrm{d}}\right) > \omega_L \left(Q_F^* \left(Q_L^{\mathrm{a}}\right), Q_L^{\mathrm{a}}\right) - \lambda f \tag{45}$$

where $\omega_L \left(0, Q_L^{\mathrm{d}}\right) = \frac{\left(1-2\sqrt{f}\right)\left(2\sqrt{f}+\lambda\right)}{(1+\lambda)^2}$ and the right hand side accounts for perceived follower entry cost.

Figure 5 plots the regions in (f, λ) space where the leader accommodates, deters, or blocks the follower's entry. The dashed lines represent the accommodation, deterrence and blockade thresholds for $\lambda = 0$. The solid black curves describe the extension of these accommodation, deterrence and blockade regions for $\lambda > 0$. If $\lambda < \sqrt{2} - 1$, between the black curves internalization drives the leader to choose deterrence ($\frac{\left(1-2\sqrt{f}\right)\left(2\sqrt{f}+\lambda\right)}{(1+\lambda)^2} > \frac{(1+\lambda)^2}{4(2+\lambda)} - \lambda f$) whereas to the left of the dashed line, the leader would accommodate without internalization. Over this region, internalization therefore shifts the leader's strategy from accommodation to deterrence.

Consumer surplus is an increasing function of total output in this model. In the figure, the parameter values where deterrence leads to greater total output (with internalization) than accommodation (without internalization) lie below the gray curve. Analytically, the

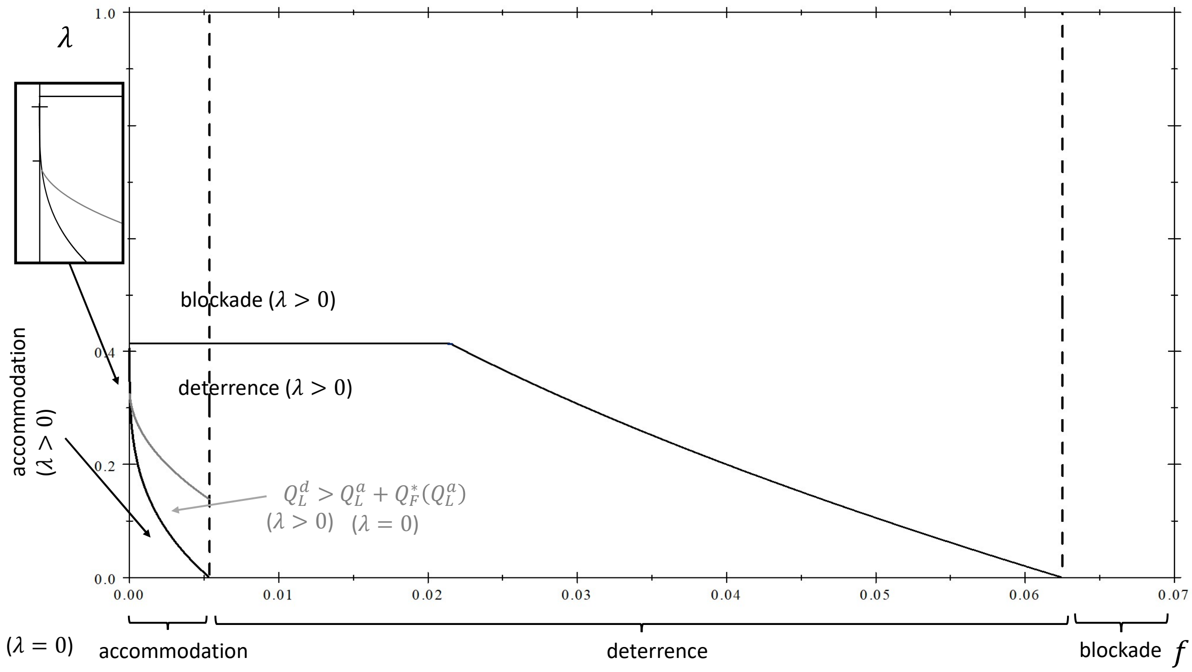

Figure 5: In the Stackelberg-Spence-Dixit model, internalization shifts the leader's strategy from accommodation to deterrence in an area delimited by the two solid black curves and the dashed line (bottom left). This strategic shift raises total capacity below the gray curve, though at very low fixed costs total capacity is always lowered (inset).

condition is $Q_L^{\mathrm{d}} = \frac{1-2\sqrt{f}}{1+\lambda} > \frac{3}{4} = \left(Q_L^{\mathrm{a}} + Q_F^*\left(Q_L^{\mathrm{a}}\right)\right)\big|_{\lambda=0}$. In the curved wedge therefore (i.e. the area delimited by the solid gray and black curves and the dashed line), the shift in the leader's strategy due to internalization raises consumer surplus. Finally, internalization leads to a jump in output under deterrence because the leader perceives part of the follower's entry cost, λf, which induces a jump in its isoprofit at the horizontal axis.

A.2 Follower value

The follower's value satisfies the asset equilibrium condition

$$r\Omega_F(X)dt = \lambda\left(1-\eta Q_L\right)Q_L X dt + E_X d\Omega_F(X). \tag{46}$$

Applying Itô's lemma and taking the expectation gives a second-order ordinary differential equation

$$r\Omega_F(X) = \lambda\left(1-\eta Q_L\right)Q_L X + \mu X\Omega_F'(X) + \frac{1}{2}\sigma^2 X^2 \Omega_F''(X). \tag{47}$$

Over the inaction region $(0, X_F^*)$ the boundary conditions are

$$\Omega_F(0) = 0 \tag{48}$$

and

$$\Omega_F\left(X_F^*\right) = G_F\left(X_F^*\right). \tag{49}$$

Moreover the optimal threshold X_F^* satisfies the smooth pasting condition

$$\Omega_F'\left(X_F^*\right) = G_F'\left(X_F^*\right). \tag{50}$$

To satisfy Eq. (47) and the lower boundary condition, conjecture a solution of the form $\Omega_F(X) = \lambda\left(1-\eta Q_L\right)Q_L \frac{X}{r-\mu} + A_F X^\beta$ where $\beta > 1$ is the upper root of $\frac{1}{2}\sigma^2 b(b-1) + \mu b - r = 0$ (see Eq. 9 in the text).

Substituting into the two remaining conditions yields

$$\begin{cases} \lambda\left(1-\eta Q_L\right) Q_L \frac{X_F^*}{r-\mu} + A_F X_F^{*\beta} = \frac{\left(1-\eta(1-\lambda) Q_L\right)^2}{4\eta} \frac{X_F^*}{r-\mu} - \frac{\delta}{2\eta}\left(1-\eta\left(1+\lambda\right) Q_L\right) + \frac{\delta^2}{4\eta} \frac{r-\mu}{X_F^*} \\ \lambda\left(1-\eta Q_L\right) Q_L \frac{1}{r-\mu} + \beta A_F X_F^{*\beta-1} = \frac{\left(1-\eta(1-\lambda) Q_L\right)^2}{4\eta} \frac{1}{r-\mu} - \frac{\delta^2}{4\eta} \frac{r-\mu}{X_F^{*2}}. \end{cases} \tag{51}$$

The first equation gives an expression for $A_F X_F^{*\beta}$. Substituting into the second and rearranging yields a quadratic,

$$(\beta-1)\left(1-\eta\left(1+\lambda\right) Q_L\right)^2 \left(\frac{X_F^*}{\delta\left(r-\mu\right)}\right)^2 - 2\beta\left(1-\eta\left(1+\lambda\right) Q_L\right) \frac{X_F^*}{\delta\left(r-\mu\right)} + \beta + 1 = 0, \tag{52}$$

of which the upper root $X_F^* = \frac{(\beta+1)\delta(r-\mu)}{(\beta-1)(1-\eta(1+\lambda)Q_L)}$ is consistent with positive capacity investment. Substituting into either of the preceding conditions then gives the expression for A_F in the text. $\square$

A.3 Leader payoff with strategic delay

For the first-order condition (Eq. 23), differentiating Eq. (22) gives

$$\frac{\left(1-2\eta Q_L\right) X}{r-\mu} - \delta - \frac{\beta}{X_F^*}\left(\frac{X}{X_F^*}\right)^\beta \frac{\delta\left(\lambda-\eta\left(1+\lambda\right)\left(\beta+1-\lambda\beta\right) Q_L\right)}{\eta\left(\beta^2-1\right)} \frac{\partial X_F^*}{\partial Q_L} \tag{53}$$
$$-\left(\frac{X}{X_F^*}\right)^\beta \frac{\delta\left(1+\lambda\right)\left(\beta+1-\lambda\beta\right)}{\left(\beta^2-1\right)}$$

and substituting for $\frac{\partial X_F^*}{\partial Q_L}\frac{1}{X_F^*} = \frac{\eta(1+\lambda)}{1-\eta(1+\lambda)Q_L}$ gives the condition in the text. The second-order condition is

$$\frac{\eta X}{r-\mu}\left(\left(\frac{X}{X_F^*}\right)^{\beta-1} \frac{\beta\left(1+\lambda\right)^2\left(2-\lambda-\eta\left(1+\lambda\right)\left(\beta+1-\lambda\beta\right) Q_L^{\mathrm{d}}\right)}{\left(\beta+1\right)\left(1-\eta\left(1+\lambda\right) Q_L^{\mathrm{d}}\right)} - 2\right) < 0. \tag{54}$$

Observe that $\frac{2-\lambda-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L^{\mathrm{d}}}{1-\eta(1+\lambda)Q_L^{\mathrm{d}}} = 1 + (1-\lambda)\frac{1-\beta\eta(1+\lambda)Q_L^{\mathrm{d}}}{1-\eta(1+\lambda)Q_L^{\mathrm{d}}} < 2-\lambda$. As $X < X_F^*$, the second-order condition therefore holds if

$$\frac{\beta+1}{\beta} > \frac{(1+\lambda)^2(2-\lambda)}{2} \tag{55}$$

which holds in particular if $\lambda < \sqrt{\frac{\beta+1}{\beta}} - 1$.

To establish that X_1^{d} is unique and constitutes a lower bound, we verify that $\frac{dQ_L^{\mathrm{d}}}{dX}(X_1^{\mathrm{d}}) > 0$ (uniqueness of Q_L^{d} by the second-order condition then implies that $Q_L^{\mathrm{d}}(X)$ separates $\aleph^{\mathrm{d}}$ into two distinct subregions). Differentiating Eq. (23) with respect to X gives

$$\frac{1-2\eta Q_L^{\mathrm{d}}}{r-\mu} - \left(\frac{X}{X_F^*}\right)^{\beta-1} \frac{\beta(1+\lambda)\left(1-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L^{\mathrm{d}}\right)}{(\beta+1)(r-\mu)}. \tag{56}$$

The value of this derivative at X_1^{d} is

$$\frac{1}{r-\mu} - \left(\frac{X_1^{\mathrm{d}}}{\delta(r-\mu)}\right)^{\beta-1} \frac{\beta(\beta-1)^{\beta-1}(1+\lambda)}{(\beta+1)^{\beta}(r-\mu)} = \frac{1}{r-\mu}\left(\beta\frac{\delta(r-\mu)}{X_1^{\mathrm{d}}} - (\beta-1)\right) \tag{57}$$

(using Eq. (24) to substitute for $\left(\frac{X_1^{\mathrm{d}}}{\delta(r-\mu)}\right)^{\beta-1}\frac{(\beta-1)^{\beta-1}(1+\lambda)}{(\beta+1)^{\beta}}$). Evaluating Eq. (24) at $X = \frac{\beta}{\beta-1}\delta(r-\mu)$ gives $\frac{1}{\beta-1}\left(\left(\frac{\beta}{\beta+1}\right)^{\beta}(1+\lambda)-1\right) < 0$, so $X_1^{\mathrm{d}} > \frac{\beta}{\beta-1}\delta(r-\mu)$ and therefore $\frac{dQ_L^{\mathrm{d}}}{dX}(X_1^{\mathrm{d}}) > 0$.

To establish that X_2^{d} (if finite) is an upper bound, we verify that $\frac{dQ_L^{\mathrm{d}}}{dX}(X_2^{\mathrm{d}}) < \frac{d\widehat{Q}_L}{dX}(X_2^{\mathrm{d}})$ From Eq. (23),

$$\frac{dQ_L^{\mathrm{d}}}{dX}(X_2^{\mathrm{d}}) = -\frac{1-2\eta Q_L^{\mathrm{d}} - \frac{\beta(1+\lambda)}{\beta+1}\left(1-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L^{\mathrm{d}}\right)}{X_2^{\mathrm{d}}\eta\left(\frac{\beta(1+\lambda)^2}{\beta+1}\frac{\left(2-\lambda-\eta(1+\lambda)(\beta+1-\lambda\beta)Q_L^{\mathrm{d}}\right)}{1-\eta(1+\lambda)Q_L^{\mathrm{d}}} - 2\right)} \tag{58}$$

and from Eq. (15),

$$\frac{d\widehat{Q}_L}{dX}\left(X_2^{\mathrm{d}}\right)=\frac{\beta+1}{\beta-1}\frac{\delta\left(r-\mu\right)}{\eta X_2^{\mathrm{d}2}\left(1+\lambda\right)}. \tag{59}$$

Comparing the two, we find that $\frac{d\widehat{Q}_L}{dX}\left(X_2^{\mathrm{d}}\right)>\frac{dQ_L^{\mathrm{d}}}{dX}\left(X_2^{\mathrm{d}}\right)$ if $\beta\lambda\left(2-\lambda(1+\lambda)\right)<1-\lambda$, the condition for the existence of X_2^{d}. $\square$

A.4 Proof of Proposition 3

The proposition is established by ruling out the possibility of a blockade strategy for the leader at all demand states.

We first show that if $X<X_1^{\mathrm{d}}$, the leader does not choose blockade. From Eqs. (22) and (28), blockade is more profitable than zero capacity if

$$\frac{\left(1+\lambda\right)\left(\beta-1\right)^{\beta-1}}{\left(\beta+1\right)^{\beta}}\left(\frac{X}{\delta(r-\mu)}\right)^{\beta}-\frac{\beta+1}{1+\lambda}\frac{X}{\delta(r-\mu)}+\frac{\beta+1}{\lambda}<0. \tag{60}$$

Recall that X_1^{d} is the lower root of

$$\frac{\left(1+\lambda\right)\left(\beta-1\right)^{\beta-1}}{\left(\beta+1\right)^{\beta}}\left(\frac{X}{\delta\left(r-\mu\right)}\right)^{\beta}-\frac{X}{\delta\left(r-\mu\right)}+1=0. \tag{61}$$

The left hand sides of both of the above conditions are convex in X, and decreasing at $X=\delta\left(r-\mu\right)$. There is a unique intersection at $X^{*}=\frac{1+\lambda}{\lambda}\frac{\beta+1-\lambda}{\beta-\lambda}\delta(r-\mu)$ where Eq. (60) cuts Eq. (61) from above. In order for Eq. (60) to have a lower root which is equal to or smaller than X_1^{d}, *i*) X^{*} must lie to the left of the minimum $X_0=\frac{(\beta+1)^{\frac{\beta}{\beta-1}}}{(\beta-1)\beta^{\frac{1}{\beta-1}}(1+\lambda)^{\frac{1}{\beta-1}}}\delta(r-\mu)$ at which Eq. (61) is strictly negative and *ii*) the left hand side of Eq. (61) must be nonnegative at X^{*}. *i*) holds if and only if $X^{*\beta-1}<X_0^{\beta-1}$, which gives

$$\Theta:=\frac{\beta\left(\beta-1\right)^{\beta-1}\left(1+\lambda\right)^{\beta}\left(\beta+1-\lambda\right)^{\beta-1}}{\left(\beta+1\right)^{\beta}\lambda^{\beta-1}\left(\beta-\lambda\right)^{\beta-1}}<1 \tag{62}$$

after rearrangement. On the other hand, *ii*) implies

$$1 \leq \frac{(\beta-1)^{\beta-1}(1+\lambda)^{\beta+1}(\beta+1-\lambda)^{\beta}}{(\beta+1)^{\beta+1}\lambda^{\beta-1}(\beta-\lambda)^{\beta-1}} = \frac{(1+\lambda)(\beta+1-\lambda)}{\beta(\beta+1)}\Theta \tag{63}$$

which is incompatible with $\Theta < 1$. Hence, *i*) and *ii*) cannot both hold, which implies that $X^* > X_1^{\mathrm{d}}$. Therefore, $Q_L = \frac{1}{\eta(1+\lambda)}$ is strictly suboptimal up to X_1^{d}.

Next, for $X \in \left[X_1^{\mathrm{d}}, X_2^{\mathrm{d}}\right]$, the leader's capacity choice problem has a local maximum over $\aleph^{\mathrm{d}}$ at Q_L^{d}. By continuity of $\Omega_L(Q_L, X)$ at $Q_L = \frac{1}{\eta(1+\lambda)}$ and because $\Omega_L(Q_L, X)$ is decreasing in Q_L thereafter, Q_L^{d} also constitutes a maximum for all $Q_L \geq \widehat{Q}_L(X)$.

Blockade is suboptimal for $X > X_1^{\mathrm{a}}$ (and hence $> X_2^{\mathrm{d}}$) if $\Omega_L\left(\frac{1}{\eta(1+\lambda)}, X\right)$ is lower than the payoff from accommodation. Evaluating Eq. (18) at $Q_L^{\mathrm{a}}(X)$ gives an explicit form,

$$\Omega_L(Q_L^{\mathrm{a}}, X) = \frac{(1+\lambda)^2 X}{4\eta(2+\lambda)(r-\mu)}\left(1 - \frac{\delta(r-\mu)}{X}\right)^2. \tag{64}$$

Blockade is therefore ruled out if

$$\frac{\lambda X}{\eta(r-\mu)(1+\lambda)^2} - \frac{\delta}{\eta(1+\lambda)} < \frac{(1+\lambda)^2 X}{4\eta(2+\lambda)(r-\mu)}\left(1 - \frac{\delta(r-\mu)}{X}\right)^2 \tag{65}$$

or after rearrangement,

$$\left(\frac{2-(1+\lambda)^2}{(1+\lambda)^2}\right)^2\left(\frac{X}{\delta(r-\mu)}\right)^2 - 2\frac{\lambda^3+3\lambda^2+\lambda-3}{(1+\lambda)^3}\frac{X}{\delta(r-\mu)} + 1 \tag{66}$$

which is positive as $\lambda < \sqrt{\frac{\beta+1}{\beta}} - 1$.

Last, to establish that $X_2^{\mathrm{d}} > X_1^{\mathrm{a}}$ it is enough to show that $\frac{1}{1-2\beta\lambda-\beta\lambda^2} > \frac{1}{2-(1+\lambda)^2}$ and $(\beta+1)(2+\lambda)(1-\beta\lambda) > \beta(2-(1+\lambda)^2) + 3 - \lambda^2$. The first inequality is straightforward for $\beta > 1$. The second can be written as $-(1+\beta)\lambda^2 + \left(\beta + \frac{1}{2}\right)\lambda + 1 > 0$, and the left hand side is a concave function of λ which is positive at $\lambda = 0$ and $\lambda = \sqrt{\frac{\beta+1}{\beta}} - 1$ and therefore positive for all $\lambda < \sqrt{\frac{\beta+1}{\beta}} - 1$. $\square$

A.5 Proof of Proposition 4

First take demand states $X \leq X_1^{\mathrm{d}}$. If a firm invests as a leader at such demand states, it sets $Q_L^* = 0$. The value of the preemption incentive is thus $f(X) = -(1-\lambda)\,\Omega_F(X,0)$, which is negative because $\Omega_F(X,0) > 0$. The preemption threshold therefore satisfies $X_P > X_1^{\mathrm{d}}$.

For the upper bound on X_P, suppose to begin with that X_1^{a} is finite. It is convenient to express the preemption incentive as $f(X) = (1-\lambda)\,\Omega_L(Q_L^*, X) - \left(1-\lambda^2\right) V_F(Q_L^*, X)$, where $V_F(Q_L^*, X)$ denotes the value of the follower's own assets (with timing and capacity choices $X_F^*(Q_L^*)$ and Q_F^* being those in Section 3). At X_1^{a}, $Q_L^* = Q_L^{\mathrm{d}}$ is optimal for the leader and therefore $(1-\lambda)\,\Omega_L(Q_L^{\mathrm{d}}, X)$ constitutes an upper bound for the first term of the preemption incentive. Higher leader capacity lowers the follower's residual demand, so $V_F(Q_L, X)$ is decreasing in Q_L and therefore $-\left(1-\lambda^2\right) V_F(Q_L, X)$ is increasing. Hence, $f\left(X_1^{\mathrm{a}}\right) > (1-\lambda)\,\Omega_L(Q_L^{\mathrm{a}}, X_1^{\mathrm{a}}) - \left(1-\lambda^2\right) V_F(Q_L^{\mathrm{a}}, X_1^{\mathrm{a}}) = \Omega_L(Q_L^{\mathrm{a}}, X_1^{\mathrm{a}}) - (\Omega_F(Q_L^{\mathrm{a}}, X_1^{\mathrm{a}}) - \lambda\delta Q_L^{\mathrm{a}})$ where the right-hand side is the value of the preemption incentive if leader capacity were suboptimally set to Q_L^{a}. At $Q_L = Q_L^{\mathrm{a}}$ however, investments are simultaneous with the follower acting as a Stackelberg quantity follower, so payoffs are those of the static Stackelberg game with internalization (see Appendix A.1). The leader's perceived payoff is therefore higher than the follower's $(\Omega_L(Q_L^{\mathrm{a}}, X_1^{\mathrm{a}}) > \Omega_F(Q_L^{\mathrm{a}}, X_1^{\mathrm{a}}) - \lambda\delta Q_L^{\mathrm{a}})$, which implies that $f\left(X_1^{\mathrm{a}}\right) > 0$.

If X_1^{a} is infinite, a similar argument can be made at X_2^{d} (if X_2^{d} is finite) with $\widehat{Q}_L\left(X_2^{\mathrm{d}}\right)$ replacing Q_L^{a}. Otherwise, both X_1^{a} and X_2^{d} are infinite and in this case $Q_L^* = Q_L^{\mathrm{d}}$ for arbitrarily high demand states. The monopoly rent term in Eq. (33) is then positive for large enough X as is the relative profit term because $\lim_{X\to\infty}\left(Q_L^{\mathrm{d}}(X) - Q_F^*\right) \geq \lim_{X\to\infty}\left(\widehat{Q}_L(X) - Q_F^*\right) = \frac{1}{\eta(1+\lambda)}$, so $f(X)$ is positive for sufficiently large X.

We conclude that the preemption range is nonempty therefore, with lower bound $X_P \in \left(X_1^{\mathrm{d}}, \min\left\{X_1^{\mathrm{a}}, X_2^{\mathrm{d}}\right\}\right)$. $\square$

A.6 Proof of Proposition 5

The preemption equilibrium is characterized by two conditions, the equilibrium condition $f\left(X_P\right) = 0$ (with the restriction that X_P be the lowest root) and the first-order condition defining $Q_L^{\mathrm{d}}\left(X_P\right)$. To express these compactly, let $\widetilde{X} = \frac{X_P}{\delta(r-\mu)}$ and $\widetilde{Q} = \eta Q_L^{\mathrm{d}}$ so that

$$\widetilde{f}\left(\widetilde{X},\widetilde{Q}\right) = \left(1-\widetilde{Q}\right)\widetilde{X} - 1 - \widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta+1}}\frac{\left(1+\beta\left(1+\lambda\right)\widetilde{Q}\right)\left(1-\left(1+\lambda\right)\widetilde{Q}\right)^{\beta}}{\widetilde{Q}} = 0 \tag{67}$$

(for Eq. 33) and

$$\widetilde{g}\left(\widetilde{X},\widetilde{Q}\right) = \left(1-2\widetilde{Q}\right)\widetilde{X} - 1 - \widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}}\left(1+\lambda\right)\left(1-\left(1+\lambda\right)\left(\beta+1-\lambda\beta\right)\widetilde{Q}\right)\left(1-\left(1+\lambda\right)\widetilde{Q}\right)^{\beta-1} = 0 \tag{68}$$

(for Eq. 23).

By the implicit function theorem, the sensitivities of $\widetilde{X}$ and $\widetilde{Q}$ with respect to λ are given by

$$\frac{d\widetilde{X}}{d\lambda} = \frac{\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\frac{\partial\widetilde{g}}{\partial\lambda} - \frac{\partial\widetilde{f}}{\partial\lambda}\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}}{\frac{\partial\widetilde{f}}{\partial\widetilde{X}}\frac{\partial\widetilde{g}}{\partial\widetilde{Q}} - \frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\frac{\partial g}{\partial\widetilde{X}}} \text{ and } \frac{d\widetilde{Q}}{d\lambda} = \frac{\frac{\partial\widetilde{f}}{\partial\lambda}\frac{\partial\widetilde{g}}{\partial\widetilde{X}} - \frac{\partial\widetilde{f}}{\partial\widetilde{X}}\frac{\partial\widetilde{g}}{\partial\lambda}}{\frac{\partial\widetilde{f}}{\partial\widetilde{X}}\frac{\partial\widetilde{g}}{\partial\widetilde{Q}} - \frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\frac{\partial g}{\partial\widetilde{X}}}. \tag{69}$$

In the above expressions, the partial derivatives with respect to λ are

$$\frac{\partial\widetilde{f}}{\partial\lambda} = \widetilde{X}^{\beta}\frac{\beta\left(\beta-1\right)^{\beta-1}}{(\beta+1)^{\beta}}\left(1+\lambda\right)\widetilde{Q}\left(1-\left(1+\lambda\right)\widetilde{Q}\right)^{\beta-1} \tag{70}$$

and

$$
\begin{aligned}
\frac{\partial \widetilde{g}}{\partial \lambda} &= -\widetilde{X}^{\beta} \frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}} \left(1-(1+\lambda)\,\widetilde{Q}\right)^{\beta-2} && (71)\\
&\quad \left(\left(\beta^{2}(1-\lambda)+\beta(1-2\lambda)+1\right)(1+\lambda)^{2}\,\widetilde{Q}^{2}-(2\beta+2-3\lambda\beta)(1+\lambda)\,\widetilde{Q}+1\right).
\end{aligned}
$$

To establish the proposition we determine the signs of $\frac{d\widetilde{X}}{d\lambda}$ and $\frac{d\widetilde{Q}}{d\lambda}$ at $\lambda=0$ and argue by continuity that these hold for small λ. Evaluated at $\lambda=0$, the system Eqs. (67, 68) becomes

$$
\begin{cases}
\left(1-\widetilde{Q}\right)\widetilde{X}-1-\widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta+1}}\frac{\left(1+\beta\widetilde{Q}\right)\left(1-\widetilde{Q}\right)^{\beta}}{\widetilde{Q}}=0\\
\left(1-2\widetilde{Q}\right)\widetilde{X}-1-\widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}}\left(1-(\beta+1)\,\widetilde{Q}\right)\left(1-\widetilde{Q}\right)^{\beta-1}=0.
\end{cases}
\tag{72}
$$

We first establish that a solution to this system satisfies $\widetilde{Q}<\frac{1}{\beta+1}$ and $\widetilde{X}<\frac{\widehat{X}_L}{\delta(r-\mu)}$. At $\widetilde{X}=\frac{X_1^{\mathrm{d}}}{\delta(r-\mu)}$ the optimal leader capacity is $Q_L^{\mathrm{d}}=0$, whereas the first-order condition for Q_L^{d} (second line in Eq. 72) implies that $\widetilde{X}=\frac{1}{1-2\widetilde{Q}}=\frac{\widehat{X}_L}{\delta(r-\mu)}>\frac{X_1^{\mathrm{d}}}{\delta(r-\mu)}$ is the only demand state at which the optimal leader capacity is $Q_L^{\mathrm{d}}=\frac{1}{\beta+1}$. Continuity of $Q_L^{\mathrm{d}}(X)$ then implies that $\widetilde{Q}<\frac{1}{\beta+1}$ for any $\widetilde{X}<\frac{\widehat{X}_L}{\delta(r-\mu)}$. Furthermore, at $\widetilde{X}=\frac{\widehat{X}_L}{\delta(r-\mu)}$ (hence $\widetilde{Q}=Q_L^{\mathrm{d}}=\frac{1}{\beta+1}$), the preemption incentive (the first line in Eq. 72) is positive, since[7]

$$
\begin{aligned}
\left(1-\widetilde{Q}\right)\widetilde{X}-1-\widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta+1}}\frac{\left(1+\beta\widetilde{Q}\right)\left(1-\widetilde{Q}\right)^{\beta}}{\widetilde{Q}} &> 0\\
&\Leftrightarrow \frac{2\beta+1}{\beta+1}\left(\frac{\beta}{\beta+1}\right)^{\beta}<1 \quad (73)
\end{aligned}
$$

Therefore, $X_P<\widehat{X}_L$, implying that a solution to Eq. (72) satisfies $\widetilde{Q}<\frac{1}{\beta+1}$.

[7] To verify the last inequality, denote the left-hand side by $\Lambda(\beta)$. Then $\Lambda(1)=.75<1$, and $\Lambda'(\beta)=\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\left(2+(2\beta+1)\ln\left(\frac{\beta}{\beta+1}\right)\right)$. Taking the first terms of the Maclaurin series, $\ln\left(\frac{\beta}{\beta+1}\right)<-\frac{1}{\beta+1}-\frac{1}{2(\beta+1)^{2}}-\frac{1}{3(\beta+1)^{3}}$, and substituting back and simplifying yields $\Lambda'(\beta)<-\frac{1}{6}\frac{\beta^{\beta}(\beta-1)}{(\beta+1)^{\beta+4}}<0$.

It is useful to note that at a solution $\left(\widetilde{X}, \widetilde{Q}\right)$ to Eq. (72),

$$\widetilde{X} = \frac{(\beta^2+\beta+1)\widetilde{Q}^2 - 2\widetilde{Q} + 1}{\left(1-\widetilde{Q}\right)\left((\beta^2+1)\widetilde{Q}^2 - 3\widetilde{Q} + 1\right)} \tag{74}$$

and

$$\widetilde{X}^{\beta} \frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta+1}} \left(1-\widetilde{Q}\right)^{\beta-1} = \frac{\widetilde{Q}^2}{\left(1-\widetilde{Q}\right)\left((\beta^2+1)\widetilde{Q}^2 - 3\widetilde{Q} + 1\right)}. \tag{75}$$

For $\lambda = 0$, the partial derivatives above (Eqs. 70 and 71) become

$$\frac{\partial \widetilde{f}}{\partial \lambda} = \widetilde{X}^{\beta} \frac{\beta\,(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}} \widetilde{Q}\left(1-\widetilde{Q}\right)^{\beta-1} \tag{76}$$

and

$$\frac{\partial \widetilde{g}}{\partial \lambda} = -\widetilde{X}^{\beta} \frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}} \left(\left(\beta^2+\beta+1\right)\widetilde{Q}^2 - 2\,(\beta+1)\,\widetilde{Q} + 1\right)\left(1-\widetilde{Q}\right)^{\beta-2}. \tag{77}$$

Using Eqs. (74) and (75) to substitute for $\widetilde{X}$ and $\widetilde{X}^{\beta}$ gives

$$\frac{\partial \widetilde{f}}{\partial \widetilde{X}} = \frac{\left(1-\widetilde{Q}\right)^2\left(1-(\beta+1)\widetilde{Q}\right)}{(\beta^2+\beta+1)\widetilde{Q}^2 - 2\widetilde{Q} + 1} > 0 \tag{78}$$

and

$$\frac{\partial \widetilde{f}}{\partial \widetilde{Q}} = \frac{(\beta+1)\widetilde{Q}}{(\beta^2+1)\widetilde{Q}^2 - 3\widetilde{Q} + 1} > 0. \tag{79}$$

Next observe that

$$\begin{aligned} \frac{\partial \widetilde{g}}{\partial \widetilde{X}} &= 1 - 2\widetilde{Q} - \widetilde{X}^{\beta-1} \frac{\beta\,(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}} \left(1-(\beta+1)\,\widetilde{Q}\right)\left(1-\widetilde{Q}\right)^{\beta-1} \\ &= \frac{\beta}{\widetilde{X}} - (\beta-1)\left(1-2\widetilde{Q}\right) = \frac{\partial \widetilde{f}}{\partial \widetilde{X}} + (\beta-1)\,\widetilde{Q} > 0 \end{aligned} \tag{80}$$

where the second line uses $\widetilde{g}\left(\widetilde{X},\widetilde{Q}\right)=0$ to substitute for the last term. Using Eqs. (74) and (75) once again to substitute for $\widetilde{X}$ and $\widetilde{X}^{\beta}$,

$$\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}=-\frac{(\beta^3-\beta-2)\widetilde{Q}^3+6\widetilde{Q}^2-6\widetilde{Q}+2}{\left(1-\widetilde{Q}\right)^2\left((\beta^2+1)\widetilde{Q}^2-3\widetilde{Q}+1\right)}<0 \tag{81}$$

(which is the leader's capacity SOC).

We show first that the denominator of $\frac{d\widetilde{X}}{d\lambda}$ and $\frac{d\widetilde{Q}}{d\lambda}$ is negative. Using Eq. (80), this can be expressed as $\frac{\partial\widetilde{f}}{\partial\widetilde{X}}\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}-\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\frac{\partial g}{\partial\widetilde{X}}=\frac{\partial\widetilde{f}}{\partial\widetilde{X}}\left(\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}-\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\right)-\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\left(\beta-1\right)\widetilde{Q}$. Because $\frac{\partial\widetilde{f}}{\partial\widetilde{X}}$ and $\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}$ are positive, it is enough to show that $\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}-\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}$ is negative. The sign of $\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}-\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}$ is that of the cubic expression $-(\beta^3-1)\widetilde{Q}^3+2(\beta-2)\widetilde{Q}^2-(\beta-5)\widetilde{Q}-2$, and this is negative because the quadratic part $2(\beta-2)\widetilde{Q}^2-(\beta-5)\widetilde{Q}-2$ is itself negative for $\widetilde{Q}<\frac{1}{\beta+1}$.

The proposition is therefore established if we show that the numerators of $\frac{d\widetilde{X}}{d\lambda}$ and $\frac{d\widetilde{Q}}{d\lambda}$ are positive. Starting with the latter, using Eq. (80) to substitute for $\frac{\partial\widetilde{g}}{\partial\widetilde{X}}$ gives $\frac{\partial\widetilde{f}}{\partial\lambda}\frac{\partial\widetilde{g}}{\partial\widetilde{X}}-\frac{\partial\widetilde{f}}{\partial\widetilde{X}}\frac{\partial\widetilde{g}}{\partial\lambda}=\frac{\partial\widetilde{f}}{\partial\widetilde{X}}\left(\frac{\partial\widetilde{f}}{\partial\lambda}-\frac{\partial\widetilde{g}}{\partial\lambda}\right)+\frac{\partial\widetilde{f}}{\partial\lambda}\left(\beta-1\right)\widetilde{Q}$. Because $\frac{\partial\widetilde{f}}{\partial\widetilde{X}}$ and $\frac{\partial\widetilde{f}}{\partial\lambda}$ are positive, it is enough to show that $\frac{\partial\widetilde{f}}{\partial\lambda}-\frac{\partial\widetilde{g}}{\partial\lambda}$ is also positive. Evaluating gives

$$\frac{\partial\widetilde{f}}{\partial\lambda}-\frac{\partial\widetilde{g}}{\partial\lambda}=\widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}}\left(1-\widetilde{Q}\right)^{\beta-2}\left((\beta^2+1)\widetilde{Q}^2-(\beta+2)\widetilde{Q}+1\right). \tag{82}$$

The sign is that of the last bracket. Express this term as a quadratic in β, $\widetilde{Q}^2\beta^2-\widetilde{Q}\beta+\left(1-\widetilde{Q}\right)^2$, which is positive for $\beta=1$ and has a negative discriminant $\widetilde{Q}^2\left(1-4\left(1-\widetilde{Q}\right)^2\right)$ because $\widetilde{Q}<.5$, so $\frac{\partial\widetilde{f}}{\partial\lambda}-\frac{\partial\widetilde{g}}{\partial\lambda}>0$.

The other numerator is

$$\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\frac{\partial\widetilde{g}}{\partial\lambda}-\frac{\partial\widetilde{f}}{\partial\lambda}\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}=\widetilde{X}^{\beta}\frac{(\beta-1)^{\beta-1}}{(\beta+1)^{\beta}}\left(1-\widetilde{Q}\right)^{\beta-2}$$
$$\times\left(-\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\left(\left(\beta^{2}+\beta+1\right)\widetilde{Q}^{2}-2\left(\beta+1\right)\widetilde{Q}+1\right)-\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}\beta\widetilde{Q}\left(1-\widetilde{Q}\right)\right).$$

The sign is that of the last bracketed term. Developing and simplifying by $\widetilde{Q}\left(1-\widetilde{Q}\right)$ leaves a cubic expression, $\left(\beta^4+\beta^3+\beta^2+1\right)\widetilde{Q}^3-\left(\beta^3+4\beta^2+3\right)\widetilde{Q}^2+\left(2\beta^2-\beta+3\right)\widetilde{Q}+\beta-1$. This is greater than $-\left(\beta^3+4\beta^2+3\right)\widetilde{Q}^2+\left(2\beta^2-\beta+3\right)\widetilde{Q}+\beta-1$, which takes the value $\beta-1>0$ at $\widetilde{Q}=0$ and $\frac{\left(2\beta^2+1\right)(\beta-1)}{(\beta+1)^2}>0$ at $\widetilde{Q}=\frac{1}{\beta+1}$, and is hence positive for all possible $\widetilde{Q}\in\left[0,\frac{1}{\beta+1}\right]$ implying $\frac{\partial\widetilde{f}}{\partial\widetilde{Q}}\frac{\partial\widetilde{g}}{\partial\lambda}-\frac{\partial\widetilde{f}}{\partial\lambda}\frac{\partial\widetilde{g}}{\partial\widetilde{Q}}>0$. $\square$